Phase Portrait

A phase portrait shows a series of trajectories for different initial conditions. The trajectories partition the plane and so will never cross each other.

Example question

Sketch the phase portrait given the following information. λ_1, λ_2 are the two eigenvalues and $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are the corresponding eigenvectors.

1. $\lambda_1 = 2,\ \lambda_2 = -1,\ \boldsymbol{p}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix},\ \boldsymbol{p}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

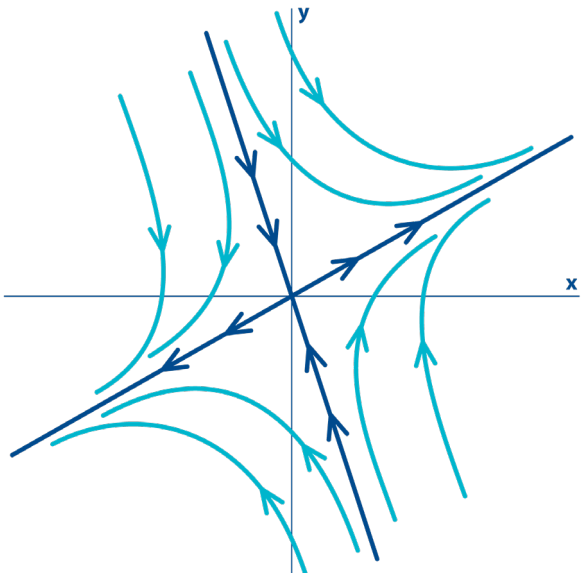
2. $\lambda_1 = 2,\ \lambda_2 = 1,\ \boldsymbol{p}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \boldsymbol{p}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

3. For $\begin{aligned} \dot{x} &= 2x - 4y \\ \dot{y} &= 5x - 2y \end{aligned}$ the eigenvalues are $\lambda_1 = 4\mathrm{i},\ \lambda_2 = -4\mathrm{i}$.

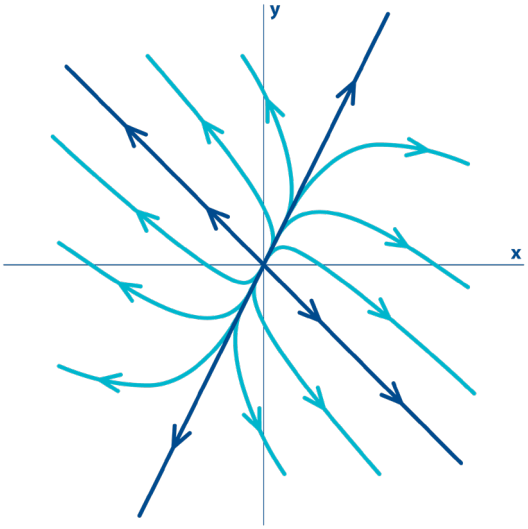
4. For $\begin{aligned} \dot{x} &= 3x - 4y \\ \dot{y} &= 8x - 5y \end{aligned}$ the eigenvalues are $\lambda_1 = -1 + 4\mathrm{i},\ \lambda_2 = -1 - 4\mathrm{i}$.

Solution

1.



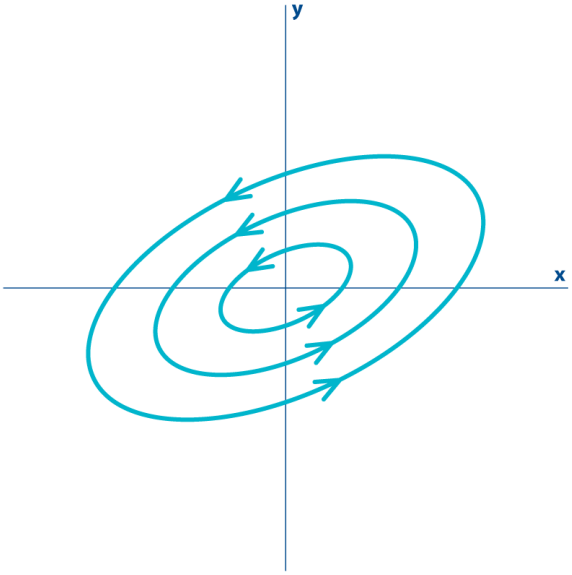
2.



3. The trajectories will be elliptical. Consider a point on one of the axes, for example $(1,0)$

$\dot{y} = 5$ hence the spiral is counterclockwise.

$\dot{x} = 4$ hence the gradient at that point is $\frac{5}{4} = 1.25$.



4. The trajectories will form spirals. Because the real part is negative they will approach $(0,0)$.

   To find the direction of the spirals consider a point on the *x* (or *y*) axis.

   For example $(2,0)$.

   At this point $\dot{y} = 16$ and hence the trajectories are counter clockwise.