1.2 EXPONENTS

♦ THE EXPONENTIAL 2^x

Let us define the power 2^x, as x moves along the sets

$N = \{0,1,2,3,\dots\}$	**Natural numbers**
$Z = \{\dots,-3,-2,-1,0,1,2,3,\dots\}$	**Integers**
$Q = \{\text{fractions } \frac{m}{n} \mid m,n\in Z,\ n\neq 0\}$	**Rational numbers**
$R = Q$ + irrational numbers[1]	**Real numbers**

1) If $x=n\in N$, then

$$2^0 = 1$$
$$2^n = 2\cdot 2\cdot 2\cdots 2 \text{ (n times)}$$

For example $2^3=8$

2) If $x=-n$, where $n\in N$, then

$$2^{-n} = \frac{1}{2^n}$$

Thus we know 2^x for any $x\in Z$.

For example $2^{-3}=\frac{1}{2^3}=\frac{1}{8}$

3) If $x=\frac{m}{n}$, where $m,n\in Z$, $n\neq 0$, then

$$2^{\frac{m}{n}} = \sqrt[n]{2^m}$$

Thus we know 2^x for any $x\in Q$

For example, $2^{\frac{2}{3}}=\sqrt[3]{2^2}=\sqrt[3]{4}$, $2^{\frac{3}{2}}=\sqrt{2^3}=\sqrt{8}$, $2^{\frac{1}{2}}=\sqrt{2}$

[1] That is numbers that cannot be expressed as fractions, eg π, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$

4) If x=irrational, then

2^x = given by a calculator!

The definition is beyond our scope, thus we trust technology!

Thus we know 2^x for any $x \in R$. For example, $2^\pi = 8.8249779$

In general, if $a>0$ we define

$a^0 = 1$

$a^n = a \cdot a \cdot a \cdots a$ (n times)

$a^{-n} = \dfrac{1}{a^n}$

$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \sqrt[n]{a}^{\,m}$

a^x = given by a calculator! (for any $x \in R$)

EXAMPLE 1

- $5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$
- $\left(\dfrac{1}{5}\right)^{-2} = \dfrac{1}{5^{-2}} = 5^2 = 25$
- $\left(\dfrac{3}{5}\right)^{-2} = \left(\dfrac{5}{3}\right)^{2} = \dfrac{25}{9}$
- $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$ or $8^{2/3} = (2^3)^{2/3} = 2^{3\cdot(2/3)} = 2^2 = 4$
- $27^{-4/3} = \sqrt[3]{27^{-4}} = \sqrt[3]{\dfrac{1}{27^4}} = \sqrt[3]{\left(\dfrac{1}{27}\right)^4} = \sqrt[3]{\left(\dfrac{1}{27}\right)}^{\,4} = \left(\dfrac{1}{3}\right)^4 = \dfrac{1}{81}$

NOTICE

If $a<0$, a^x is defined only for $x=n \in Z$	• $0^x=0$ only if $x \neq 0$ • 0^0 is not defined

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- PROPERTIES

All known properties of powers are still valid for exponents $x \in R$

(1) $a^x a^y = a^{x+y}$

(2) $\dfrac{a^x}{a^y} = a^{x-y}$

(3) $(ab)^x = a^x b^x$

(4) $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

(5) $(a^x)^y = a^{xy}$

Here $a,b>0$ and $x,y \in R$

EXAMPLE 2

Express the following as single powers (i.e. in the form x^y)

expression	your answer	correct answer
a^3a^2		a^5
$\dfrac{a^6}{a^2}$		a^4
$\dfrac{x^3x^5}{x^4}$		x^4
$2^{x+1}2^{3x}$		2^{4x+1}
$8x^3$		$2^3x^3 = (2x)^3$
$\dfrac{x^3y^3}{z^3}$		$\left(\dfrac{xy}{z}\right)^3$
$\dfrac{16a^2}{b^4}$		$\dfrac{4^2a^2}{b^4} = \left(\dfrac{4a}{b^2}\right)^2$
$(x^3)^4$		x^{12}

- THE NUMBER e

There is a specific irrational number

$$e=2.7182818...$$

which plays an important role in mathematics, especially in exponential modelling which we are going to study later. The number e is almost as popular as the irrational number π=3.14…

An approximation of e is given below. Consider the expression

$$\left(1+\frac{1}{n}\right)^n$$

For n=1	the result is	2
For n=2	the result is	2.25
For n=10	the result is	2.5937424…
For n=100	the result is	2.7048138…
For n=1000	the result is	2.7169239…
For $n=10^6$	the result is	2.7182804…

As n tends to +∞ this expression tends to e=2.7182818…

EXAMPLE 3

Express the following as single powers of e (i.e. in the form e^a)

expression	your answer	correct answer
$(e^2)^3e^3$		$e^6e^3 = e^9$
$e^{x+1}e^{3x}$		e^{4x+1}
$(e^x)^2$		e^{2x}
$\frac{e^xe^3}{e^4}$		e^{x-1}
$(\frac{1}{e})^{3x}$		e^{-3x}

♦ SIMPLE EXPONENTIAL EQUATIONS

If $a\neq1$, then

$$a^x=a^y \Rightarrow x=y$$

EXAMPLE 4

Solve the following equations

(a) $2^{3x-1} = 2^{x+2}$ (b) $2^{3x-1} = 4^{x+2}$ (c) $4^{3x-1} = 8^{x+2}$

(d) $\frac{1}{2^{3x-1}} = 4^{x+2}$ (e) $\sqrt{2}^{\,3x-1} = 4^{x+2}$

Solution

Attempt to induce a common base on both sides

(a) We have already a common base. Thus

$$2^{3x-1} = 2^{x+2} \Leftrightarrow 3x-1 = x+2 \Leftrightarrow 2x = 3 \Leftrightarrow \boxed{x=3/2}$$

(b) We can write $4=2^2$. Thus

$$2^{3x-1} = 4^{x+2} \Leftrightarrow 2^{3x-1} = 2^{2x+4} \Leftrightarrow 3x-1 = 2x+4 \Leftrightarrow \boxed{x = 5}$$

(c) We can write $4=2^2$ and $8=2^3$. Thus

$$4^{3x-1} = 8^{x+2} \Leftrightarrow 2^{6x-2} = 2^{3x+6} \Leftrightarrow 6x-2 = 3x+6$$

$$\Leftrightarrow 3x=8 \Leftrightarrow \boxed{x = 8/3}$$

(d) We apply the property $\frac{1}{2^n}=2^{-n}$. Thus

$$\frac{1}{2^{3x-1}} = 4^{x+2} \Leftrightarrow 2^{-3x+1} = 2^{2x+4} \Leftrightarrow -3x+1 = 2x+4$$

$$\Leftrightarrow 5x = -3 \Leftrightarrow \boxed{x=-3/5}$$

(e) We apply the property $\sqrt{2}=2^{1/2}$. Thus

$$\sqrt{2}^{\,3x-1} = 4^{x+2} \Leftrightarrow 2^{\frac{3x-1}{2}} = 2^{2x+4} \Leftrightarrow \frac{3x-1}{2} = 2x+4$$

$$\Leftrightarrow 3x-1= 4x+8 \Leftrightarrow \boxed{x=-9}$$
