1.2 EXPONENTS

♦ THE EXPONENTIAL 2^x

Let us define the power 2^x, as x moves along the sets

N = {0,1,2,3,…}	**Natural numbers**
Z = {…,−3,−2,−1,0,1,2,3,…}	**Integers**
Q = {fractions $\frac{m}{n}$ | m,n∈Z, n≠0}	**Rational numbers**
R = Q + irrational numbers[1]	**Real numbers**

1) If x=n∈N, then

$$2^0 = 1$$
$$2^n = 2\cdot 2\cdot 2\cdots 2 \text{ (n times)}$$

For example $2^3=8$

2) If x=−n, where n∈N, then

$$2^{-n} = \frac{1}{2^n}$$

Thus we know 2^x for any x∈Z.

For example $2^{-3}=\frac{1}{2^3}=\frac{1}{8}$

3) If $x=\frac{m}{n}$, where m,n∈Z, n≠0, then

$$2^{\frac{m}{n}} = \sqrt[n]{2^m}$$

Thus we know 2^x for any x∈Q

For example, $2^{\frac{2}{3}}=\sqrt[3]{2^2}=\sqrt[3]{4}$, $2^{\frac{2}{3}}=\sqrt{2^3}=\sqrt{8}$, $2^{\frac{1}{2}}=\sqrt{2}$

[1] That is numbers that cannot be expressed as fractions, eg π, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$

4) If x=irrational, then

2^x = given by a calculator!

The definition is beyond our scope, thus we trust technology!

Thus we know 2^x for any $x \in R$. For example, $2^\pi = 8.8249779$

In general, if a>0 we define

$a^0 = 1$

$a^n = a \cdot a \cdot a \cdots a$ (n times)

$a^{-n} = \frac{1}{a^n}$

$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \sqrt[n]{a}^{\,m}$

a^x = given by a calculator! (for any $x \in R$)

EXAMPLE 1

- $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$
- $\left(\frac{1}{5}\right)^{-2} = \frac{1}{5^{-2}} = 5^2 = 25$
- $\left(\frac{3}{5}\right)^{-2} = \left(\frac{5}{3}\right)^2 = \frac{25}{9}$
- $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$ or $8^{2/3} = (2^3)^{2/3} = 2^{3\cdot(2/3)} = 2^2 = 4$
- $27^{-4/3} = \sqrt[3]{27^{-4}} = \sqrt[3]{\frac{1}{27^4}} = \sqrt[3]{\left(\frac{1}{27}\right)^4} = \sqrt[3]{\left(\frac{1}{27}\right)}^{\,4} = \left(\frac{1}{3}\right)^4 = \frac{1}{81}$

NOTICE

If a<0, a^x is defined only for $x = n \in Z$	• $0^x = 0$ only if $x \neq 0$ • 0^0 is not defined

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- PROPERTIES

All known properties of powers are still valid for exponents $x \in R$

(1) $a^x a^y = a^{x+y}$

(2) $\dfrac{a^x}{a^y} = a^{x-y}$

(3) $(ab)^x = a^x b^x$

(4) $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

(5) $(a^x)^y = a^{xy}$

Here $a,b>0$ and $x,y \in R$

EXAMPLE 2

Express the following as single powers (i.e. in the form x^y)

expression	your answer	correct answer
a^3a^2		a^5
$\dfrac{a^6}{a^2}$		a^4
$\dfrac{x^3x^5}{x^4}$		x^4
$2^{x+1}2^{3x}$		2^{4x+1}
$8x^3$		$2^3x^3 = (2x)^3$
$\dfrac{x^3y^3}{z^3}$		$\left(\dfrac{xy}{z}\right)^3$
$\dfrac{16a^2}{b^4}$		$\dfrac{4^2a^2}{b^4} = \left(\dfrac{4a}{b^2}\right)^2$
$(x^3)^4$		x^{12}

♦ THE NUMBER e

There is a specific irrational number

$$e=2.7182818\ldots$$

which plays an important role in mathematics, especially in exponential modelling which we are going to study later. The number e is almost as popular as the irrational number π=3.14…

An approximation of e is given below. Consider the expression

$$\left(1+\frac{1}{n}\right)^n$$

For n=1	the result is	2
For n=2	the result is	2.25
For n=10	the result is	2.5937424…
For n=100	the result is	2.7048138…
For n=1000	the result is	2.7169239…
For $n=10^6$	the result is	2.7182804…

As n tends to +∞ this expression tends to e=2.7182818…

EXAMPLE 3

Express the following as single powers of e (i.e. in the form e^a)

expression	your answer	correct answer
$(e^2)^3e^3$		$e^6e^3 = e^9$
$e^{x+1}e^{3x}$		e^{4x+1}
$(e^x)^2$		e^{2x}
$\dfrac{e^xe^3}{e^4}$		e^{x-1}
$(\frac{1}{e})^{3x}$		e^{-3x}

♦ SIMPLE EXPONENTIAL EQUATIONS

If $a\neq 1$, then

$$a^x=a^y \Rightarrow x=y$$

EXAMPLE 4

Solve the following equations

(a) $2^{3x-1} = 2^{x+2}$ (b) $2^{3x-1} = 4^{x+2}$ (c) $4^{3x-1} = 8^{x+2}$

(d) $\frac{1}{2^{3x-1}} = 4^{x+2}$ (e) $\sqrt{2}^{\,3x-1} = 4^{x+2}$

Solution

Attempt to induce a common base on both sides

(a) We have already a common base. Thus

$$2^{3x-1} = 2^{x+2} \Leftrightarrow 3x-1 = x+2 \Leftrightarrow 2x = 3 \Leftrightarrow \boxed{x=3/2}$$

(b) We can write $4=2^2$. Thus

$$2^{3x-1} = 4^{x+2} \Leftrightarrow 2^{3x-1} = 2^{2x+4} \Leftrightarrow 3x-1 = 2x+4 \Leftrightarrow \boxed{x = 5}$$

(c) We can write $4=2^2$ and $8=2^3$. Thus

$$4^{3x-1} = 8^{x+2} \Leftrightarrow 2^{6x-2} = 2^{3x+6} \Leftrightarrow 6x-2 = 3x+6$$

$$\Leftrightarrow 3x=8 \Leftrightarrow \boxed{x = 8/3}$$

(d) We apply the property $\frac{1}{2^n}=2^{-n}$. Thus

$$\frac{1}{2^{3x-1}} = 4^{x+2} \Leftrightarrow 2^{-3x+1} = 2^{2x+4} \Leftrightarrow -3x+1 = 2x+4$$

$$\Leftrightarrow 5x = -3 \Leftrightarrow \boxed{x=-3/5}$$

(e) We apply the property $\sqrt{2} = 2^{1/2}$. Thus

$$\sqrt{2}^{\,3x-1} = 4^{x+2} \Leftrightarrow 2^{\frac{3x-1}{2}} = 2^{2x+4} \Leftrightarrow \frac{3x-1}{2} = 2x+4$$

$$\Leftrightarrow 3x-1= 4x+8 \Leftrightarrow \boxed{x=-9}$$

1.3 SEQUENCES IN GENERAL – SERIES

♦ SEQUENCE

A **sequence** is just an ordered list of numbers (**terms** in a definite order). For example

2,	5,	13,	5,	−4,	…
↑	↑	↑	↑	↑	
1st term	2nd term	3rd term	4th term	5th term	

Usually, the terms of a sequence follow a specific pattern, for example

0,2,4,6,8,10,… (even numbers)

1,3,5,7,9,11,… (odd numbers)

5,10,15,20,25,… (positive multiples of 5)

2,4,8,16,32,… (powers of 2)

We use the notation u_n to describe the n-th term. Thus, the terms of the sequence are denoted by

$$u_1, u_2, u_3, u_4, u_5, \ldots$$

♦ SERIES

A **series** is just a sum of terms:

$S_n = u_1 + u_2 + u_3 + \cdots + u_n$ (the sum of the first n terms)

$S_\infty = u_1 + u_2 + u_3 + \cdots$ (the sum of all terms, ∞ terms)

We say that S_∞ is an infinite series, while the finite sums $S_1, S_2, S_3,\ldots$ are called **partial sums**.

EXAMPLE 1

Consider the sequence

1,3,5,7,9,11,... (odd numbers)

Some of the terms are the following

$u_1=1$, $u_2=3$, $u_3=5$, $u_6=11$, $u_{10}=19$

Also,

$S_1=1$,

$S_2=1+3=4$,

$S_3=1+3+5=9$,

$S_4=1+3+5+7=16$

Finally,

$S_\infty=1+3+5+7+\cdots$ (in this case the result is $+\infty$)

♦ SIGMA NOTATION ($\sum_{n=1}^{k}$)

Instead of writing

$$u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9$$

we may write

$$\sum_{n=1}^{9}u_n$$

It stands for the sum of all terms u_n, where n ranges from 1 to 9.

In general,

$$\sum_{n=1}^{k}u_n$$

expresses the sum of all terms u_n, where n ranges from 1 to k.

We may also start with another value for n, instead of 1, e.g. $\sum_{n=4}^{9}u_n$

EXAMPLE 2

- $\sum_{n=1}^{3} 2^n = 2^1 + 2^2 + 2^3 = 2+4+8 = 14$
- $\sum_{n=1}^{4} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{12+6+4+3}{12} = \frac{25}{12}$
- $\sum_{k=1}^{3} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{4+2+1}{8} = \frac{7}{8}$
- $\sum_{n=3}^{6} (2n+1) = 7+9+11+13 = 22$
- $\sum_{x=3}^{20} \frac{x}{x+2} = \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \cdots + \frac{20}{22} =$ … whatever that is, I don't mind!!!

We can also express an infinite sum as follows

- $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ (it never finishes!)

The result is 1. (I know it looks strange, but believe me, it is right!)

♦ NOTICE

There are two basic ways to describe a sequence

A) by a GENERAL FORMULA

We just describe the general term u_n in terms of n.

For example, $u_n = 2n$ (It gives $u_1 = 2$, $u_2 = 4$, $u_3 = 6$, …)

It is the sequence 2,4,6,8,10,…

EXAMPLE 3

$u_n = n^2$ is the sequence $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

that is 1, 4, 9, 16, 25, …

$u_n = 2^n$ is the sequence 2, 4, 8, 16, 32, …

B) by a RECURSIVE RELATION (mainly for Math HL)

Given: u_1 , the first term

u_{n+1} in terms of u_n

For example,

$u_1 = 10$

$u_{n+1} = u_n + 2$

This says that the first term is 10 and then

$u_2 = u_1 + 2$

$u_3 = u_2 + 2$

$u_4 = u_3 + 2$ and so on.

In simple words, begin with 10 and keep adding 2 in order to find the following term.

It is the sequence 10, 12, 14, 16, 18, …

EXAMPLE 4

$u_1 = 3$ $u_{n+1} = 2u_n + 5$

It is the sequence 3, 11, 27, 59, …

EXAMPLE 5

Sometimes, we are given the first two terms u_1, u_2 and then a recursive formula for u_{n+1} in terms of u_n and u_{n-1}.

The most famous sequence of this form is the **Fibonacci sequence**

$u_1 = 1$, $u_2 = 1$

$u_{n+1} = u_n + u_{n-1}$

In other words,

we add u_1, u_2 in order to obtain u_3,

we add u_2, u_3 in order to obtain u_4, and so on.

It is the sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

1.4 ARITHMETIC SEQUENCE (A.S.)

♦ THE DEFINITION

Let's start with an example! I give you the first term of a sequence, say u_1=5, and I always ask you to add a fixed value, say d=3, in order to find the next term. The following sequence is generated

5, 8, 11, 14, 17, …

Such a sequence is called **arithmetic**. That is, in an arithmetic sequence the difference between any two consecutive terms is constant.

We only need

The **first term**	u_1
The **common difference**	d

EXAMPLE 1

If u_1=1, d=2 the sequence is 1, 3, 5, 7, 9, …

If u_1=2, d=2 the sequence is 2, 4, 8, 10, 12, …

If u_1=-10, d=5 the sequence is -10, -5, 0, 5, 10, …

If u_1=10, d=-3 the sequence is 10, 7, 4, 1, -2, …

Notice that the common difference d may also be negative!

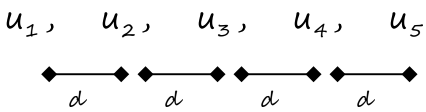
♦ **QUESTION A**: What is the general formula for u_n?

If we know u_1 and d, then

$$u_n = u_1 + (n-1)d$$

Indeed, let us think:

In order to find u_5, we start from u_1 and then add 4 times the difference d



Hence, $u_5 = u_1 + 4d$

Similarly, $u_{10} = u_1 + 9d$

$u_{50} = u_1 + 49d$

In general, $u_n = u_1 + (n-1)d$

EXAMPLE 2

In an arithmetic sequence let $u_1=3$ and $d=5$. Find

(a) the first four terms (b) the 100^{th} term

Solution

(a) 3, 8, 13, 18

(b) Now we need the general formula

$$u_{100} = u_1 + 99d = 3 + 99 \cdot 5 = 498$$

EXAMPLE 3

In an arithmetic sequence let $u_1=100$ and $u_{16} = 145$. Find u_7

Solution

We know u_1, we need d. We exploit the information for u_{16} first.

$$u_{16} = u_1 + 15d$$

$$145 = 100 + 15d$$

$$45 = 15d$$

$$d=3$$

Therefore, $u_7 = u_1 + 6d = 100 + 6 \cdot 3 = 118$

REMEMBER: Usually, our first task in an A.S. is to find the basic elements, u_1 and d, and then everything else!

EXAMPLE 4

In an arithmetic sequence let $u_{10}=42$ and $u_{19}=87$. Find u_{100}

Solution

The formula for u_{10} and u_{19} takes the form

$u_{10}=u_1+9d$ thus $u_1+9d=42$ (a)

$u_{19}=u_1+18d$ $u_1+18d=87$ (b)

Subtract (b)-(a): $18d-9d=87-42$

$9d=45$

$d=5$

Then, (a) gives $u_1=42-9d$

$=42-9\cdot5$

$=-3$

Since we know $u_1=-3$ and d = 5 we are able to find any term we like! Thus,

$$u_{100}=u_1+99d=-3+99\cdot5=492$$

♦ **QUESTION B**: What is the sum S_n of the first n terms?

It is directly given by

$$S_n=\frac{n}{2}(u_1+u_n) \qquad (1)$$

or otherwise by

$$S_n=\frac{n}{2}[2u_1+(n-1)d] \qquad (2)$$

NOTICE: Use (1) if you know u_1 and the last term u_n

Use (2) if you know u_1 and d (the basic elements)

EXAMPLE 5

For the A.S. 3, 5, 7, 9, 11, … find S_3 and S_{101}

Solution

We have $u_1=3$ and $d=2$. For S_3 the result is direct:

$$S_3 = 3+5+7 = 15$$

[check though that formulas (1), (2) give the same result for S_3]

For S_{101} we use formula (2)

$$S_{101} = \frac{101}{2}[2u_1 + 100d] = \frac{101}{2}206 = 10403$$

EXAMPLE 6

Find 10 + 20 + 30 + … + 200

Solution

We have an arithmetic sequence with $u_1=10$ and $d=10$.

The number of terms is clearly 20 and $u_{20}=200$

$$S_{20} = \frac{20}{2}(u_1 + u_{20}) = 10\ (10+200) = 2100$$

EXAMPLE 7

Show that

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Solution

This is the simplest arithmetic series with $u_1=1$ and $d=1$.

We ask for S_n

$$S_n = \frac{n}{2}(u_1 + u_n) = \frac{n}{2}(1+n) = \frac{n(n+1)}{2}$$

For example,

$$1+2+3+ \ldots + 100 = \frac{100 \cdot 101}{2} = 5050$$

EXAMPLE 8

The 3rd term of an A.S. is zero while the sum of the first 15 terms is -300. Find the first term and the sum of the first ten terms.

Solution

Well, too much information!!! Let us organize our data:

GIVEN: $u_3=0$ $S_{15}=-300$

ASK FOR: u_1 S_{10}

The formulas for u_3 and S_{15} give

$$u_3 = u_1 + 2d \quad \Leftrightarrow \quad 0 = u_1 + 2d$$

$$S_{15} = \frac{15}{2}(2u_1 + 14d) \quad \Leftrightarrow -300 = 15u_1 + 105d$$

We solve the system

$$u_1 + 2d = 0$$
$$15u_1 + 105d = -300$$

And obtain $u_1 = 8$ and $d = -4$.

Finally,

$$S_{10} = \frac{10}{2}(2u_1 + 9d) = 5(16 - 36) = -100$$

♦ NOTICE FOR CONSECUTIVE TERMS

Let

$$a, \; x, \; b$$

be consecutive terms of an arithmetic sequence (we don't mind if these are the first three terms or some other three consecutive terms). The common difference is equal to

$$x - a = b - x$$

Hence, 2x=a+b, that is $x = \frac{a+b}{2}$ (x is the mean of a and b)

EXAMPLE 9

Let x+1, 3x, 6x–5 be consecutive terms of an A.S. Find x.

Solution

It holds $(3x)-(x+1) = (6x-5)-(3x)$

$\Leftrightarrow \quad 2x-1 = 3x-5$

$\Leftrightarrow \quad x = 4$

(Indeed, the three terns are 5, 12, 19)

EXAMPLE 10

Let a, 10, b, a+b be consecutive terms of an A.S. Find a and b

Solution

Clearly $10-a = b-10 = (a+b)-b$

that is $10-a = b-10 = a$

Hence,

$10-a = a \Leftrightarrow 2a = 10 \Leftrightarrow a = 5$

$b-10 = a \Leftrightarrow b-10 = 5 \Leftrightarrow b = 15$

EXAMPLE 11

Let 100, a, b, c, 200 be consecutive terms of an A.S. Find the values of a, b and c.

Solution

Notice that 100, b, 200 are also in arithmetic sequence.

Thus b is the mean of 100 and 200, that is b=150

Now

a is the mean of 100 and 150, that is a = 125

c is the mean of 150 and 200, that is c = 175