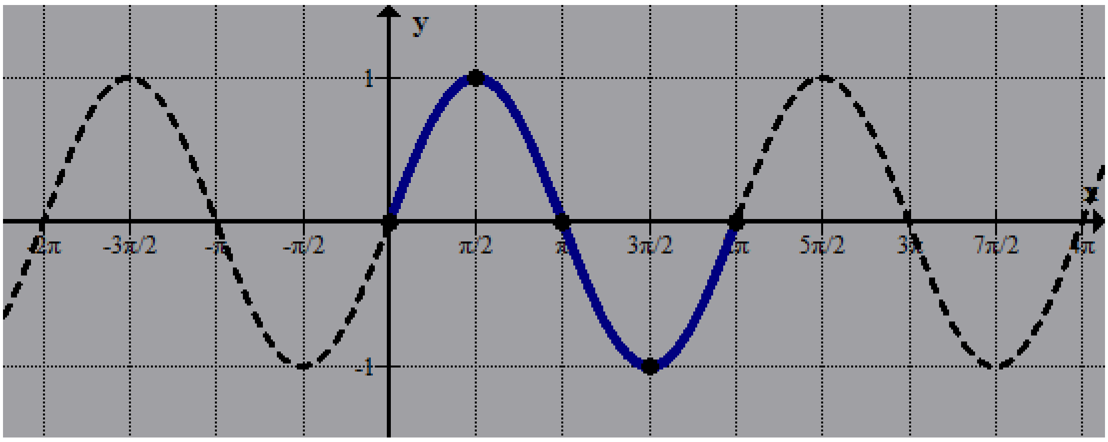
3.7 TRIGONOMETRIC FUNCTIONS

- **$f(x) = \sin x$**

Let us construct the graph of this function in the traditional way, that is in the Cartesian plane Oxy.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	...
f(x)	0	1	0	-1	0	



We have:

Domain: $x \in R$

Range: $y \in [-1,1]$ [since $y_{min}=-1$ and $y_{max}=1$]

For functions of this form we also define

Central line: $y = \frac{y_{max}+y_{min}}{2}$

Amplitude $= \frac{y_{max}-y_{min}}{2}$ (or $= y_{max}$−central value)

Period = length of a complete cycle (it is denoted by T)

Thus, for $f(x)=\sin x$

Central line: $y = 0$

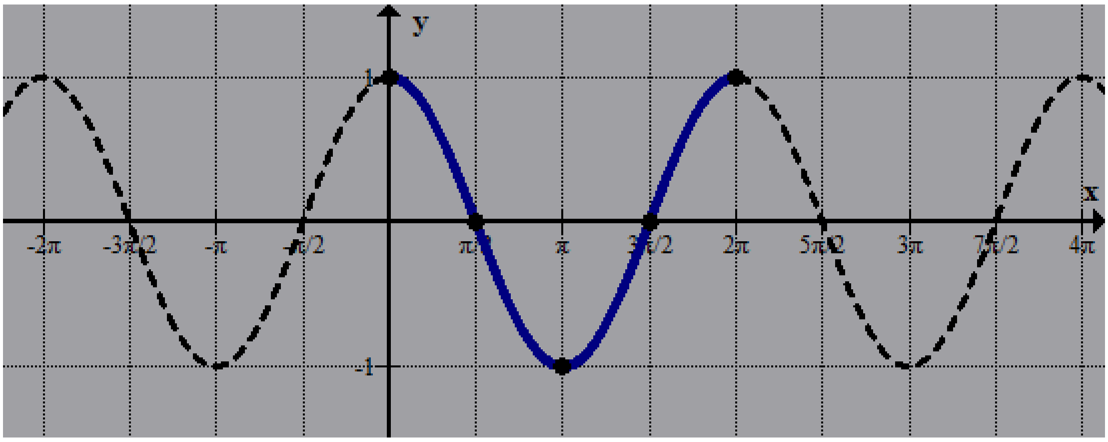
Amplitude = 1

Period: $T= 2\pi$ (that is, the graph is repeated every 2π)

- **f(x) = cosx**

Similarly, let us now construct the graph of this function.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	…
f(x)	1	0	−1	0	1	



Again we have:

Domain: $x \in R$

Range: $y \in [-1,1]$ $[y_{min}=-1$ and $y_{max}=1]$

Central line: $y = 0$

Amplitude = 1

Period: $T = 2\pi$

NOTICE:

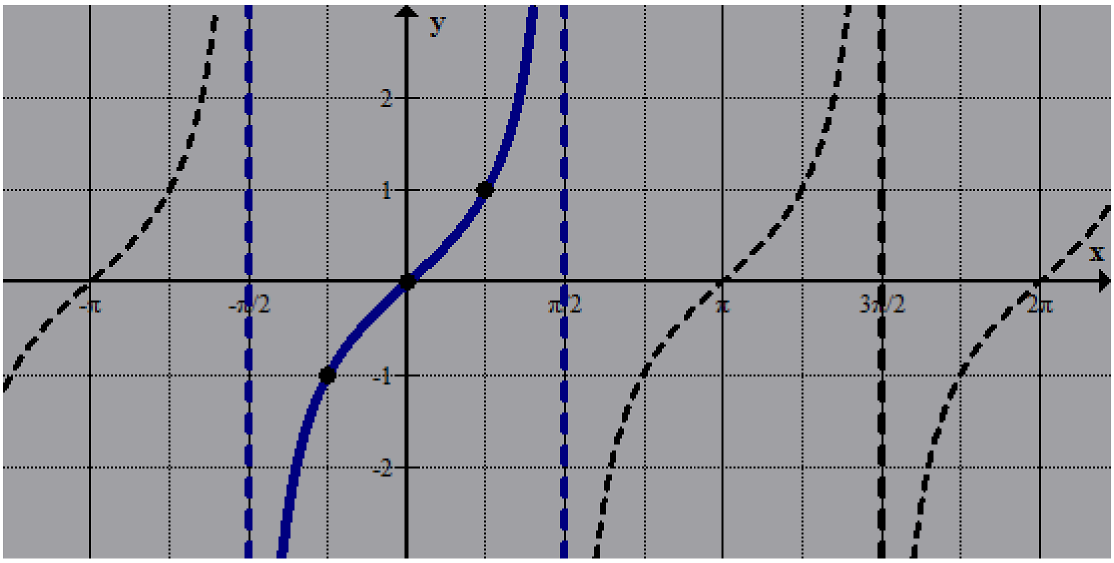
- sinx and cosx have similar graphs. The graph of cosx is a horizontal translation of sinx by $\pi/2$ units to the left.
- For both functions y=sinx and y=cosx

  distance between two consecutive max = 2π (one period)
  distance between two consecutive min = 2π (one period)
  distance between consecutive max and min = π (half a period)

♦ **f(x) = tanx**

Similarly, we construct the graph of this function.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	…
f(x)	–	-1	0	1	–	



We have:

Domain: $x \in R - \{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\}$

Range: $y \in R$ [there is no min, no max, no amplitude]

Central line: y=0

Period: T = π

Vertical asymptotes: $x=\frac{\pi}{2}$, $x=-\frac{\pi}{2}$, etc

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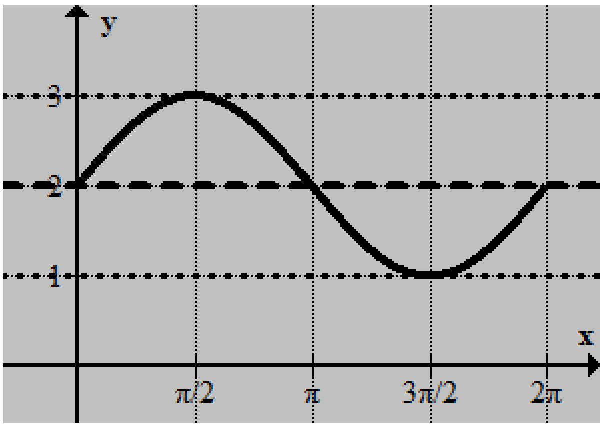
NOTICE: Remember our discussion about asymptotes in Topic 2.

It is enough to know that a vertical line x=a is an asymptote of the graph y=f(x) if

- the function is not defined at x=a, and
- for values of x very close to a the value of y=f(x) approaches $+\infty$ or $-\infty$

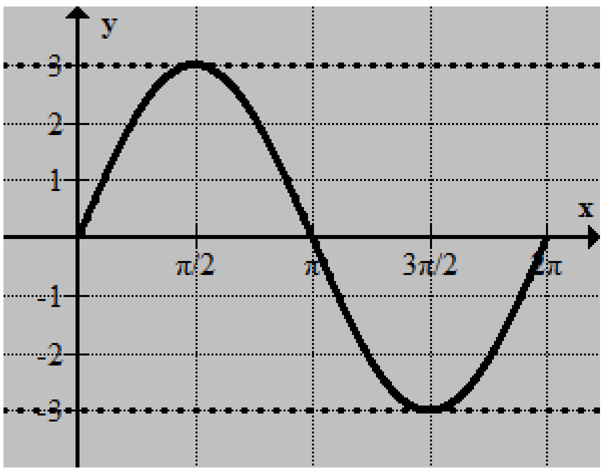
- **TRANSFORMATIONS OF sinx, cosx, tanx**

Consider the function f(x) = sinx + 2.
Its graph is a vertical translation of sinx, 2 units up:



Clearly, **Central line at y=2** (also min = 1, max = 3)
Amplitude = 1
Period: T = 2π

Consider the function f(x) = 3sinx.
Its graph is a vertical stretch of sinx with scale factor 3:

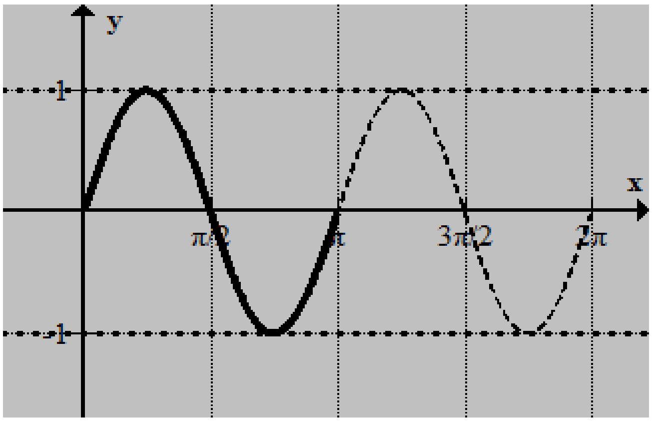


Clearly, **Central line at y=0** (also min = –3, max = 3)
Amplitude = 3
Period = 2π

Notice: the amplitude of f(x)=–3sinx is still 3

Consider $f(x) = \sin 2x$.

Its graph is a horizontal stretch of sinx with scale factor 1/2:



Now, **Central line at y=0** (also min = -1, max = 1)

Amplitude = 1

Period: T = π

In general, the function

$$f(x) = A\sin Bx + C$$

(with A>0, B>0) is a sequence of three transformations of sinx:

a vertical stretch with scale factor **A**,

a horizontal stretch with scale factor 1/**B**,

a vertical translation by **C** units (up or down),

Mind that, if **A**<0 we start with a reflection in x-axis;

Consequently,

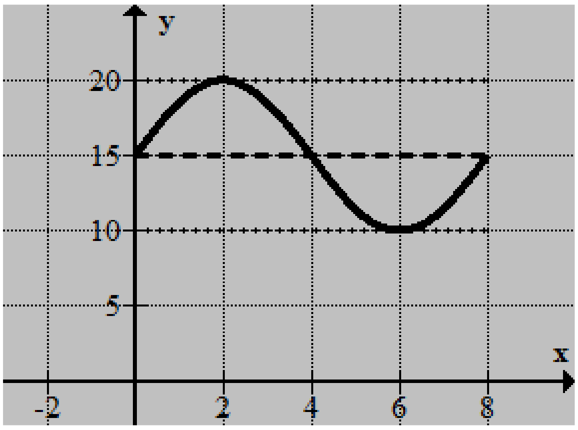
$\|A\|$	The new **amplitude**
C	The new **central value**. The central line is **y=C**
$T=\frac{2\pi}{B}$	The new **period**. Hence $B=\frac{2\pi}{T}$

Notice:

- f(x) ranges between the values $C \pm A$
- Similar observations apply for **f(x) = AcosBx +C**

EXAMPLE 1

The graph of f(x) = AsinBx + C is given below. Find A,B,C.

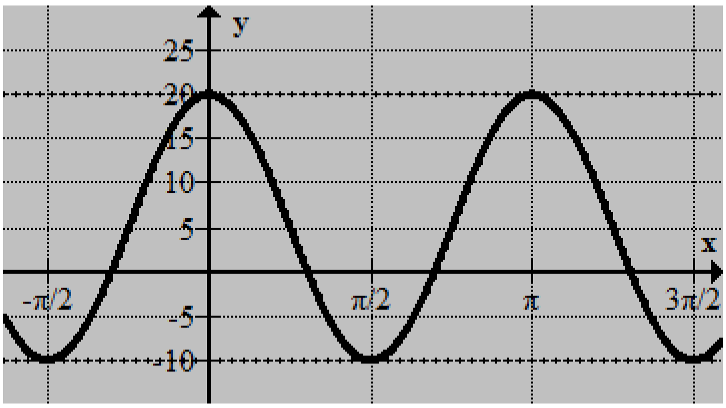


- Central line at y=15, so C=15
- Amplitude = 5, so A=5
- Period T= 8, hence $B=\frac{2\pi}{T}=\frac{2\pi}{8}=\frac{\pi}{4}$

Therefore, the equation of the function is **$f(x) = 5\sin(\frac{\pi}{4}x) + 15$**

EXAMPLE 2

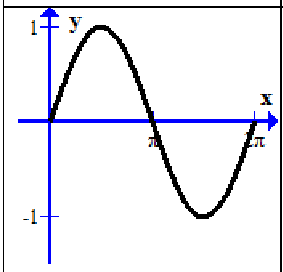
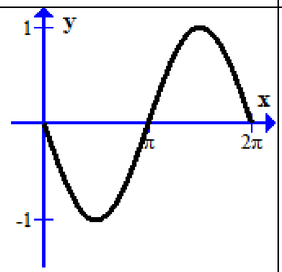
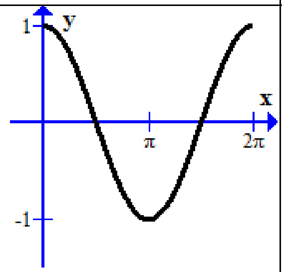
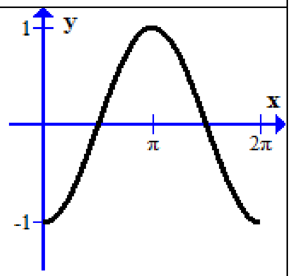
The graph of f(x) = AcosBx +C is given below. Find A,B,C.



- Central line at 5, so C=5 (since $\frac{max+min}{2}=\frac{20-10}{2}=5$)
- Amplitude = 15, so A=15 (since max-C =15)
- Period T= π, hence $B=\frac{2\pi}{T}=\frac{2\pi}{\pi}=2$

Therefore, the equation of the function is **$f(x) = 15\cos(2x) + 5$**

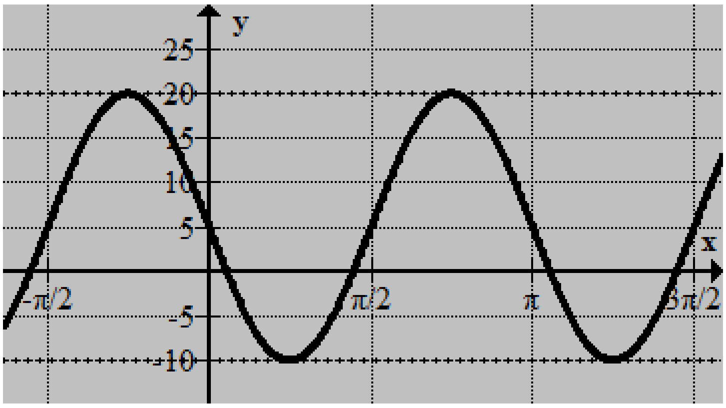
Mind the following basic types of trigonometric functions:

sinx	−sinx	cosx	−cosx
			
y-intercept central going up	y-intercept central going down	y-intercept max	y-intercept min

Notice that the amplitude is always positive but the coefficient A of sinx or cosx can be positive or negative.

EXAMPLE 3

Express the following graph as a trigonometric function.



We can easily find that

Central line: $y=5$ hence $C=5$

Amplitude = 15

Period: $T=\pi$ hence $B=\frac{2\pi}{\pi}=2$

The function of type −sinx (y-int central going down), so $A=-15$

Therefore, the equation of the function is

$$f(x) = -15\sin(2x) + 5$$

Conversely, if we are given a trigonometric function we can easily draw the graph

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EXAMPLE 4

Draw the graph of the function $f(x) = 5\sin 2x+7, \quad 0 \le x \le 2\pi$

Solution

Central value = 7

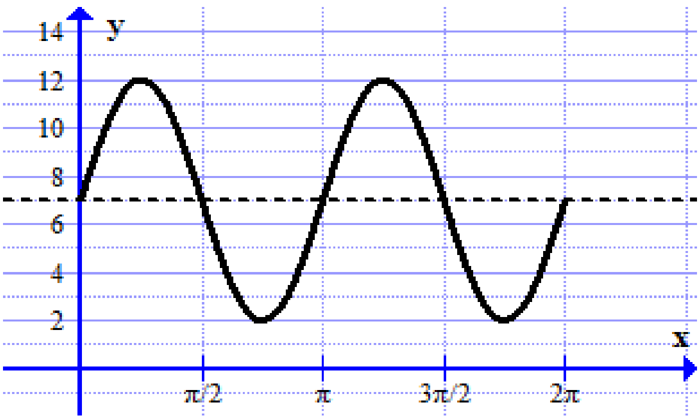
Amplitude = 5

max=12, min=2 (since f(x) ranges between 7±5)

Period $T = \frac{2\pi}{2} = \pi$

Thus, we have to draw two periods. The starting point, that is the y-intercept, is on the central line (going up).

The graph is



Finally, remember the horizontal transformations f(x-a):

sin(x-D) cos(x-D)	translation D units **to the right**

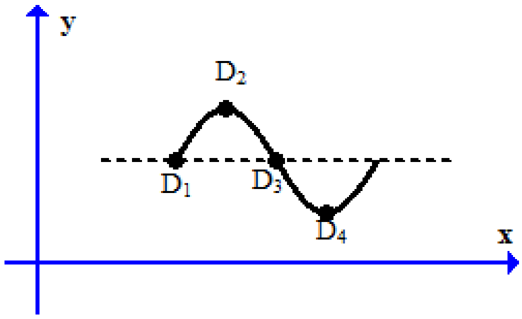
Therefore, for the functions

$f(x) = A\sin[B(x-D)] + C$ $\qquad$ $f(x) = A\cos[B(x-D)] + C$

A,B,C are determined as above.

D shows a horizontal translation.

Practically, for the value of D, whenever we see a graph like



D_1 for sinx

D_2 for cosx

D_3 for -sinx

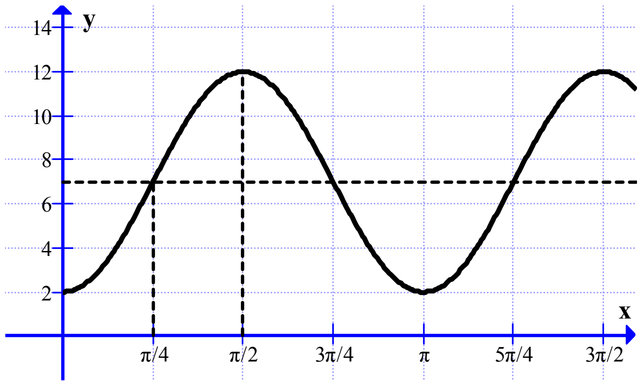
D_4 for -cosx

we spot the points shown above!

The x-coordinate of the appropriate point is the value of D.

EXAMPLE 5

Consider the graph of a trigonometric function given below.



Central value = 7,	thus C =7
Amplitude = 5,	thus $\|A\|=5$
Period T = π ,	thus B=2π/π = 2

As it starts from a min, the most appropriate form (without D) is

$$f(x)= -5\cos 2x+7$$

However, the same function can be expressed as

$$f(x)= 5\sin[2(x-\frac{\pi}{4})]+7$$

$$f(x)= 5\cos[2(x-\frac{\pi}{2})]+7$$

$$f(x)= -5\sin[2(x-\frac{3\pi}{4})]+7$$

In a similar way,

$$f(x) = A\tan Bx + C$$

is a transformation of tanx, where

$\lvert A\rvert$	Scale factor of vertical stretch
$T=\frac{\pi}{B}$	The new period
C	The new central value (units up or down)

EXAMPLE 6

$$f(x)= 10\tan 4x + 30$$

central value= 30

NO min, NO max (10 simply shows a vertical stretch of tanx)

Period = $\frac{\pi}{4}$

Finally,

tan(x−D)	translation D units **to the right or left**

Therefore, for the function

$$f(x) = A\tan[B(x-D)] + C$$

A,B,C are determined as above.

D shows a horizontal translation.