2.9 LOGARITHMS – THE LOGARITHMIC FUNCTION $y=\log_a x$

♦ THE LOGARITHM $\log_2 x$

This number is called **logarithm of x to the base 2**. It is connected to the exponential 2^x. The definition is given by

$$\log_2 x = y \Leftrightarrow 2^y = x$$

For example,

$\log_2 8 = 3$, since $2^3=8$

$\log_2 16 = 4$, since $2^4=16$

$\log_2 1024 = 10$, since $2^{10}=1024$

etc.

For example, for $\log_2 8=?$, we think in the following way:

$2^{\text{what exponent}}$ gives 8?

The answer is 3

Hence $\log_2 8=3$

Working in the same way let us find $\log_2 64=?$

It is $\log_2 64=6$

However, for $\log_2 10=?$, we should think in the following way:

$2^{\text{what exponent}}$ gives 10?

OK, this is difficult to answer!!!

Our calculator gives $\log_2 10=3.321928\ldots$

This implies that

$$2^{3.321928\ldots}=10$$

EXAMPLE 1

Find $\log_2 32$, $\log_2 2^5$, $\log_2 2^{100}$, $\log_2 2^{1453}$, $\log_2 2$, $\log_2 1$

- $\log_2 32=5$
- $\log_2 2^5=5$
- $\log_2 2^{100}=100$ Notice, in general $\mathbf{\log_2 2^x=x}$
- $\log_2 2^{1453}=1453$
- $\log_2 2=1$
- $\log_2 1=0$

♦ THE LOGARITHM $\log_a x$

In exactly the same way, for any base $a>0$, $a\neq 1$ we define

$$\log_a x = y \Leftrightarrow a^y = x$$

For example, $\log_3 9 = 2$ (since $3^2=9$)

NOTICE

Once upon a time $\log_{10} x$ has been the most popular logarithm!!!

$\log_{10} 1000 = 3$, $\log_{10} 10000 = 4$, $\log_{10} 1000000 = 6$,

$\log_{10} 0.001 = -3$, $\log_{10} 0.000001 = -6$,

In some way, the logarithm to the base 10 indicates the size of the number! Due to its popularity the base 10 for this particular logarithm is usually ignored

We write **$\log x$** instead of **$\log_{10} x$**

Hence $\log 100=2$ $\log 0.01=-2$

♦ THE LOGARITHMIC FUNCTION $y=log_a x$

A new function is defined

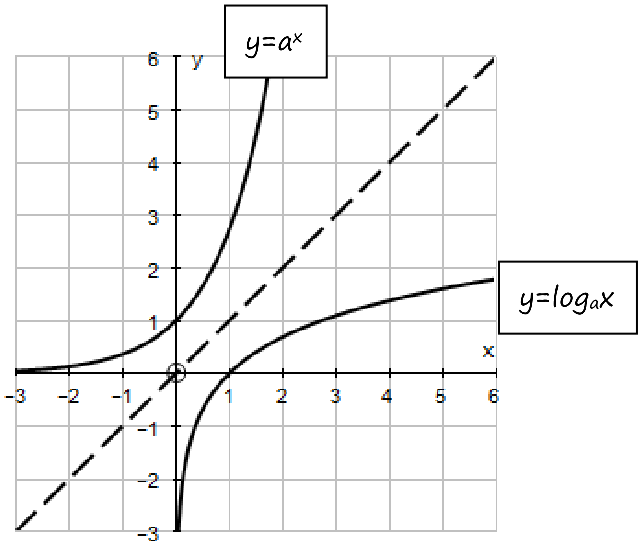
$$y = log_a x$$

In fact, this is the inverse function of the exponential function $y=a^x$

If $f(x)=a^x$ then $f^{-1}(x)=log_a x$

Indeed, $a^x=y \Leftrightarrow x= log_a y$, hence $f^{-1}(x)=log_a x$

If a>1 (for example if a=2), the graphs of these two functions look like



Observations:

- For $y=a^x$: **Domain**: $x \in R$ **Range**: $y \in R_+$ (i.e. $y>0$)
- For $y=log_a x$: **Domain**: $x \in R_+$ (i.e. $x>0$) **Range**: $y \in R$
- The x-axis is a horizontal asymptote of $y=a^x$
- The y-axis is a vertical asymptote of $y=log_a x$
- $y=a^x$ always passes through (0,1)
- $y=log_a x$ always passes through (1,0)

♦ BASIC PROPERTIES OF LOGARITHMS

For any base a ($a>0$, $a\neq 1$)

- $\log_a 1 = 0$
- $\log_a a = 1$
- $\log_a a^x = x$
- $a^{\log_a x} = x$

The first three results can be directly confirmed by the definition of logarithm. For the last one, set $y=\log_a x$. The definition implies $a^y=x$. Replace back $y=\log_a x$ and the result is immediate!

♦ FOUR ALGEBRAIC LAWS

For simplicity reasons, we use log instead of $\log_a$.

1) $\log xy = \log x + \log y$
2) $\log \frac{x}{y} = \log x - \log y$
3) $\log x^n = n\log x$
4) $\log \frac{1}{x} = -\log x$

or

- $\log x + \log y = \log xy$
- $\log x - \log y = \log \frac{x}{y}$
- $n\log x = \log x^n$
- $-\log x = \log \frac{1}{x}$

Proofs (consider all logarithms to be of base a)

For all of them we follow the same method! We check if $a^{LHS} = a^{RHS}$

1) $a^{LHS} = xy$ and $a^{RHS} = a^{\log_a x+\log_a y} = a^{\log_a x}a^{\log_a y} = xy$

2) $a^{LHS} = x/y$ and $a^{RHS} = a^{\log_a x-\log_a y} = a^{\log_a x}/a^{\log_a y} = x/y$

3) $a^{LHS} = x^n$ and $a^{RHS} = a^{n\log_a x} = (a^{\log_a x})^n = x^n$

4) this is a special case of 2) if we set $x=1$, as well as of 3) if $n=-1$

NOTICE

1) The first two laws can be combined in the following way:

$$\log A + \log B - \log C + \log D = \log \frac{ABD}{C}$$

If we also have coefficients we can work as in the following example

$$\mathbf{2}\log A + \mathbf{3}\log B - \mathbf{4}\log C + \mathbf{5}\log D = \log A^2 + \log B^3 - \log C^4 + \log D^5$$

$$= \log \frac{A^2B^3D^5}{C^4}$$

Thus

$$\mathbf{2}\log A + \mathbf{3}\log B - \mathbf{4}\log C + \mathbf{5}\log D = \log \frac{A^2B^3D^5}{C^4}$$

This is the way **<u>we convert many logs into one log</u>**. It is useful when we solve equations [see Example 2) below]

Look at also the opposite direction

$$\log \frac{A^2B^3D^5}{C^4} = \mathbf{2}\log A + \mathbf{3}\log B - \mathbf{4}\log C + \mathbf{5}\log D$$

This is the way **<u>we express one log in terms of many logs</u>**. It is useful when we need to simplify logarithms [see Example 3) below].

2) When we solve equations involving logs our target is to collect all logs together and convert them into one log.

- If we obtain $\mathbf{\log_a A(x) = \log_a B(x)}$

  then we get $\mathbf{A(x)=B(x)}$ (we simply eliminate logs)

- If we obtain $\mathbf{\log_a A(x) = c}$

  then we get $\mathbf{A(x)=a^c}$ by definition

In both cases the new equation is much simpler, without logs.

EXAMPLE 2

Solve the equations

(a) $\log_2 x+\log_2(x+2)= \log_2 3$

(b) $\log_2 x+\log_2(x+2)= 3$

(c) $\log_2 x+\log_2(x-2)-\log_2(x-\frac{3}{4})=\log_2 3$

Solutions

(a) We obtain $\log_2 x(x+2)= \log_2 3$

Hence

$$x(x+2)=3 \Leftrightarrow x^2+2x-3=0$$

The solutions are $x=1$ and $x=-3$

The second solution is rejected since $x>0$ and $x+2>0$ by the original equation. Therefore $\boxed{x=1}$.

(b) We obtain $\log_2 x(x+2)= 3$

Hence

$$x(x+2)=2^3 \Leftrightarrow x^2+2x-8=0$$

The solutions are $x=2$ and $x=-4$

The second solution is rejected since $x>0$ and $x+2>0$ by the original equation. Therefore $\boxed{x=2}$.

(c) We obtain $\log_2 \frac{x(x-2)}{(x-\frac{3}{4})} = \log_2 3$

Hence

$$\frac{x(x-2)}{(x-\frac{3}{4})}=3 \Leftrightarrow x^2-2x=3x-\frac{9}{4} \Leftrightarrow x^2-5x+\frac{9}{4}=0$$

The solutions are $x=4.5$ or $x=0.5$

The second solution is rejected. Therefore, $\boxed{x=4.5}$

- THE NATURAL LOGARITHM **lnx**

The most frequently used logarithm is the logarithm to the base

$$e=2.7182818...$$

Instead of $\log_e x$, we denote it by

$$\ln x$$

Hence,

$$\ln x=y \Leftrightarrow e^y=x$$

- CHANGE OF BASE

Consider the equation

$$a^x=b$$

If you apply $\log_a$ on both sides you obtain

$$x=\log_a b$$

If you apply any other logarithm, say log, ln, $\log_5$, $\log_c$ you obtain

$$x=\frac{\log b}{\log a}, \quad x=\frac{\ln b}{\ln a}, \quad x=\frac{\log_5 b}{\log_5 a}, \quad x=\frac{\log_c b}{\log_c a}$$

respectively. Thus

$$\log_a b = \frac{\log b}{\log a} = \frac{\ln b}{\ln a} = \frac{\log_c b}{\log_c a}$$

That is, we can change $\log_a b$ into $\frac{\log_* b}{\log_* a}$, in any base we like.

The formula

$$\log_a b=\frac{\log_c b}{\log_c a}$$

is known as the "**change of base formula**".

For example

$$\log_2 5 = \frac{\log 5}{\log 2} = \frac{0.699}{0.301} = 2.322 \quad \text{or} \quad \frac{\ln 5}{\ln 2} = \frac{1.609}{0.693} = 2.322$$

Let us see an example where we need to express one logarithm in terms of many logarithms

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EXAMPLE 3

Suppose $\ln x=\mathbf{a}$, $\ln y=\mathbf{b}$, $\ln z=\mathbf{c}$, as well as $\ln 2=\mathbf{m}$, $\ln 5=\mathbf{n}$.
Express the following in terms of **a, b, c, m, n**.

$\ln xy$, $\ln x^2$, $\ln\frac{y}{z}$, $\ln\frac{x^3y}{z^2}$, $\ln\frac{1}{x}$, $\ln\sqrt{x}$,

$\ln 10$, $\ln 50$, $\ln 2.5$, $\log_2 x$, $\log_5 e$, $\log_4 x^3$

Solution

- $\ln xy = \ln x + \ln y = \mathbf{a+b}$
- $\ln x^2 = 2\ln x = \mathbf{2a}$
- $\ln\frac{y}{z} = \ln y - \ln z = \mathbf{b-c}$
- $\ln\frac{x^3y}{z^2} = 3\ln x + \ln y - 2\ln z = \mathbf{3a+b-2c}$
- $\ln\frac{1}{x} = \ln 1 - \ln x = 0-a = \mathbf{-a}$ [or $\ln\frac{1}{x} = \ln x^{-1} = -\ln x = \mathbf{-a}$]
- $\ln\sqrt{x} = \ln x^{1/2} = \frac{1}{2}\ln x = \mathbf{\frac{a}{2}}$
- $\ln 10 = \ln(2\times 5) = \ln 2 + \ln 5 = \mathbf{m+n}$
- $\ln 50 = \ln(2\times 5^2) = \ln 2 + 2\ln 5 = \mathbf{m+2n}$
- $\ln 2.5 = \ln\frac{5}{2} = \ln 5 - \ln 2 = \mathbf{n-m}$
- $\log_2 x = \frac{\ln x}{\ln 2} = \mathbf{\frac{a}{m}}$
- $\log_5 e = \frac{\ln e}{\ln 5} = \mathbf{\frac{1}{n}}$
- $\log_4 x^3 = 3\log_4 x = 3\frac{\ln x}{\ln 4} = 3\frac{\ln x}{\ln 2^2} = \mathbf{\frac{3a}{2m}}$

2.10 EXPONENTIAL EQUATIONS

In these equations the unknown x is in the exponent. The simplest exponential equation has the form

$$a^x=b$$

If we apply $\log_a$ the solution is $x=\log_a b$

If we apply log or ln the solution is $x=\frac{\log b}{\log a}$ or $x=\frac{\ln b}{\ln a}$

EXAMPLE 1

Solve the equation $2(5^x) = 9$. Express the result in the form $\frac{\log a}{\log b}$.

Solution

We first divide by 2 and then apply log

$$5^x = 4.5 \Leftrightarrow \log 5^x = \log 4.5 \Leftrightarrow x\log 5 = \log 4.5 \Leftrightarrow \boxed{x = \frac{\log 4.5}{\log 5}}$$

Notice

If we use ln( ), the answer will be $\boxed{x = \frac{\ln 4.5}{\ln 5}}$

If we use $\log_5$( ), the answer will be $\boxed{x = \log_5 4.5}$

Whenever we see exponentials of base e, it is preferable to use ln( ).

EXAMPLE 2

Solve the equation $10e^{2x} = 85$

Solution

We first divide by 10:

$$10e^{2x} = 85 \Leftrightarrow e^{2x} = 8.5 \Leftrightarrow \ln e^{2x} = \ln 8.5 \Leftrightarrow 2x = \ln 8.5 \Leftrightarrow \boxed{x= \frac{\ln 8.5}{2}}$$

EXAMPLE 3

Solve the equation $5^x = 2^{x+1}$. Express the result in the form $\frac{\ln a}{\ln b}$.

Solution

Method A: Let us apply ln on both sides

$$5^x = 2^{x+1} \Leftrightarrow \ln 5^x = \ln 2^{x+1}$$

$$\Leftrightarrow x\ln 5 = (x+1)\ln 2$$

$$\Leftrightarrow x\ln 5 = x\ln 2 + \ln 2$$

$$\Leftrightarrow x\ln 5 - x\ln 2 = \ln 2$$

$$\Leftrightarrow x(\ln 5 - \ln 2) = \ln 2$$

$$\Leftrightarrow x = \frac{\ln 2}{\ln 5 - \ln 2} \Leftrightarrow \boxed{x = \frac{\ln 2}{\ln \frac{5}{2}}}$$

Method B: Simplify the equation to the form $a^x=b$; then apply ln

$$5^x = 2^{x+1} \Leftrightarrow 5^x = 2^x 2$$

$$\Leftrightarrow \frac{5^x}{2^x} = 2$$

$$\Leftrightarrow \left(\frac{5}{2}\right)^x = 2$$

$$\Leftrightarrow x\ln\left(\frac{5}{2}\right) = \ln 2$$

$$\Leftrightarrow \boxed{x = \frac{\ln 2}{\ln \frac{5}{2}}}$$

Remarks

- This is the exact answer. If we are looking for an answer to 3sf, the calculator gives x=0.756.
- We can use any logarithm instead of ln( ), for example log( ).
- If an expression in the form $\log_a b$ is required, the answer is

$$\left(\frac{5}{2}\right)^x = 2 \Leftrightarrow x = \log_{\frac{5}{2}} 2$$