Differential equations, phase portraits and Euler's method

Mathematics: applications and interpretation sections AHL5.16, AHL5.17 and AHL5.18

Exact solutions to systems of coupled linear differential equations

The system of equations $\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned}$ can be written as $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

which in turn can be written as $\dot{\boldsymbol{x}} = \boldsymbol{M}\boldsymbol{x}$ where the dot notation indicates a variable has been differentiated with respect to *t.*

If λ_1 and λ_2 are the two eigenvalues of ***M*** (and the course content requires that these are distinct) and $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are the corresponding eigenvectors, then the solution to the system of equations is:

$\boldsymbol{x} = Ae^{\lambda_1 t}\boldsymbol{p}_1 + Be^{\lambda_2 t}\boldsymbol{p}_2$ where the constants *A* and *B* will depend on the initial conditions for the system and this is given in the formula booklet.

Verifying that this is a solution to the original equation can be demonstrated by differentiating each side, and using the properties of eigenvalues and eigenvectors.

Differentiating gives:

$$\dot{\boldsymbol{x}} = A\lambda_1 e^{\lambda_1 t}\boldsymbol{p}_1 + B\lambda_2 e^{\lambda_2 t}\boldsymbol{p}_2 = Ae^{\lambda_1 t}\left(\lambda_1\boldsymbol{p}_1\right) + Be^{\lambda_2 t}\left(\lambda_2\boldsymbol{p}_2\right).$$

Using the property of eigenvectors that $\boldsymbol{M}\boldsymbol{p} = \lambda\boldsymbol{p}$ we can rewrite the equation as

$\dot{\boldsymbol{x}} = Ae^{\lambda_1 t}\boldsymbol{M}\boldsymbol{p}_1 + Be^{\lambda_2 t}\boldsymbol{M}\boldsymbol{p}_2 = \boldsymbol{M}\left(Ae^{\lambda_1 t}\boldsymbol{p}_1 + Be^{\lambda_2 t}\boldsymbol{p}_2\right) = \boldsymbol{M}\boldsymbol{x}$ and hence is a solution to the differential equation.

Real eigenvalues

If the eigenvalues are real, students should be able to find and interpret a particular solution to the equation.

The course specifies finding exact solutions in the case of real distinct eigenvalues only. Students will not be asked to find particular solutions in the case of complex or imaginary eigenvalues, as this is not covered in the course content.

Example question

a. Find the general solution of the following system of equations.

b. Find the particular solution given that $x = 4,\ y = -5$ when $t = 0$,

Solution

a. $M = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(-4-\lambda) - 6 = 0 \Rightarrow -4 - \lambda + 4\lambda + \lambda^2 - 6 = 0 \Rightarrow \lambda^2 + 3\lambda - 10 = 0$$

$\Rightarrow \lambda = -5$ or 2.

Hence

$$\lambda = -5 \Rightarrow \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$\Rightarrow y = -3x$ so an eigenvector is $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

and when $\lambda = 2 \Rightarrow \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\Rightarrow y = \dfrac{1}{2}x$ so the other eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

The **general solution** for the system of equations is therefore:

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-5t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + Be^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

b. To find the particular solution for the initial conditions the values given are substituted into the general equation in order to find values for *A* and *B*.

$$4 = A + 2B$$
$$-5 = -3A + B$$

$$A = 2,\ B = 1$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = 2e^{-5t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Qualitative and long-term behavior

For linear systems $(0,0)$ will always be an equilibrium point, defined as a point at which the derivative is equal to zero for all the variables.

A **trajectory** shows the path traced out by a solution to the system of equations, as the value of *t* increases.

Real eigenvalues

For systems with real eigenvalues there are three possibilities depending on the signs of the eigenvalues.

If $\lambda_1, \lambda_2 < 0$ then each term of the equation $\boldsymbol{x} = Ae^{\lambda_1 t}\boldsymbol{p}_1 + Be^{\lambda_2 t}\boldsymbol{p}_2$ will tend to zero as $t \to \infty$ and $(0,0)$, so the origin is a **stable** equilibrium point.

In this case the trajectories will approach the $(0,0)$ along the direction of the eigenvector with the least negative eigenvalue.

If $\lambda_1, \lambda_2 > 0$ then the movement will be away from the origin with the trajectories tending towards the direction of the eigenvector with largest eigenvalue.

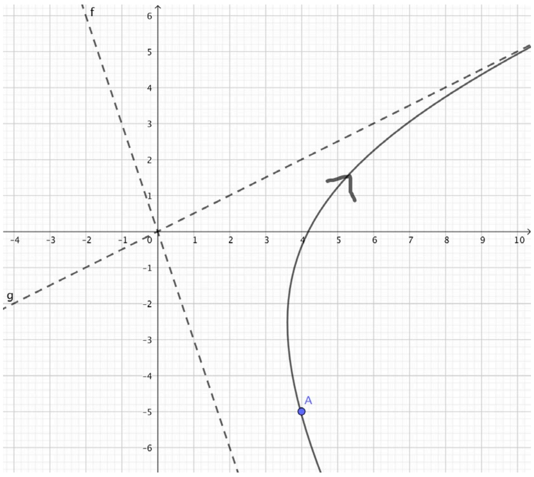
In this case $(0,0)$ is an unstable equilibrium point.

If $\lambda_1 > 0$ and $\lambda_2 < 0$ then as $t \to \infty$ the trajectories will have the line $\boldsymbol{x} = \mu\boldsymbol{p}_1,\ \mu \in \mathbb{R}$ as an asymptote and as $t \to -\infty$ the line $\boldsymbol{x} = \mu\boldsymbol{p}_2$ will be an asymptote.

In this case $(0,0)$ is a **saddle point** and as such an unstable equilibrium point.

The trajectory of the particular solution calculated above,

$\begin{pmatrix} x \\ y \end{pmatrix} = 2e^{-5t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, is shown below:



Complex eigenvalues

Because they come from the solution of a quadratic equation, complex eigenvalues will always form conjugate pairs.

If the eigenvalues are $a \pm b\mathrm{i}$ then the solution to the system is:

$\boldsymbol{x} = Ae^{(a+b\mathrm{i})t}\boldsymbol{p}_1 + Be^{(a-b\mathrm{i})t}\boldsymbol{p}_2$ which can be written as $\boldsymbol{x} = e^{at}\left(Ae^{b\mathrm{i}t}\boldsymbol{p}_1 + Be^{-b\mathrm{i}t}\boldsymbol{p}_2\right)$.

The fact that $e^{\theta\mathrm{i}} = \cos\theta + \mathrm{i}\sin\theta$ indicates that second factor introduces circular motion to the system. The first factor, $\boldsymbol{e^{at}}$ will ensure the trajectory spirals away from $(0,0)$ when $a > 0$ and towards $(0,0)$ when $a < 0$.

In exams candidates will be expected to be able to determine whether the direction of the trajectory is clockwise or counter clockwise. One way of determining this is to find the value of $\dot{y}$ as a trajectory crosses the *x*-axis or the value of $\dot{x}$ as it crosses the *y*-axis.

Example question

Sketch the trajectory of the solution to the following system of equations:

$\dot{x} = x + 5y$
$\dot{y} = -5x + y$ given that when $t = 0$, $x = 2$ and $y = 0$.

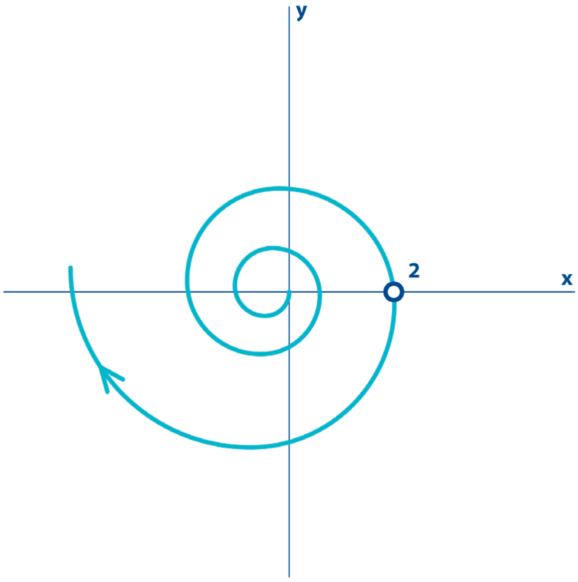
Solution

Following the same method as the previous example the eigenvalues can be calculated:

$\lambda = 1 \pm 5\mathrm{i}$

At the point $(2,0)$ $\dot{y} = -10$ and hence the spiral is moving clockwise.

The trajectory is therefore:



Imaginary eigenvalues

In this case the trajectories will form circles or ellipse with $(0,0)$ as a centre. The direction of the trajectories can be determined in a similar manner to that of the complex eigenvalues. The value of $\dot{x}$ and $\dot{y}$ can give an indication of the alignment of the ellipse.