Numerical solutions to coupled differential equations.

Most applications of coupled differential equations will involve non-linear equations and so usually cannot be solved directly.

Fortunately there are many numerical methods for finding approximate solutions, one of which is the Euler method covered in this course.

For

$$\frac{dx}{dt} = f_1(x, y, t)$$

$$\frac{dy}{dt} = f_2(x, y, t)$$

The Euler formula is

$$x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n)$$
$$y_{n+1} = y_n + h \times f_2(x_n, y_n, t_n)$$
$$t_{n+1} = t_n + h$$

Example question

Use the Euler method with a step length of 0.1 to find the value of *x* and *y* when $t = 1$, given that $x = 1,\ y = 1$ when $t = 0$.

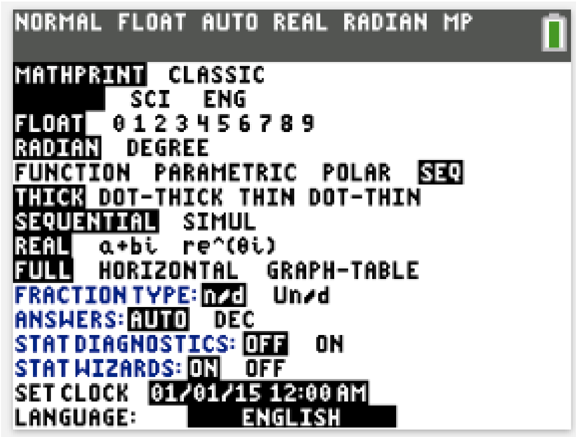
$$\frac{dx}{dt} = 3x - xy$$
$$\frac{dy}{dt} = xy - 2y$$

Solution

Values when $t = 1$ are $x = 6.5408,\ y = 1.92566$

TI-84

Put the calculator into sequence **SEQ** mode.

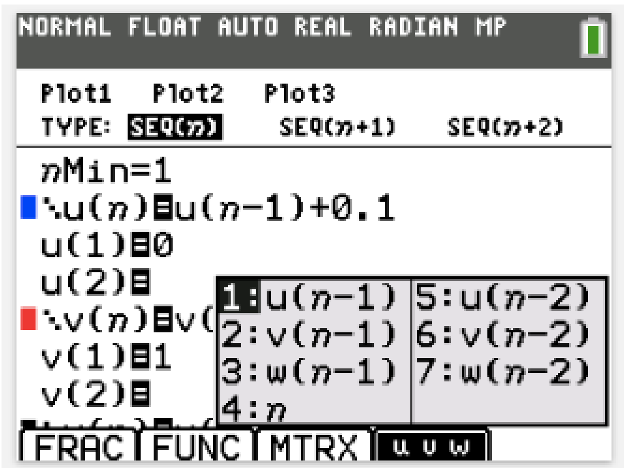


The sequences can then be added using 

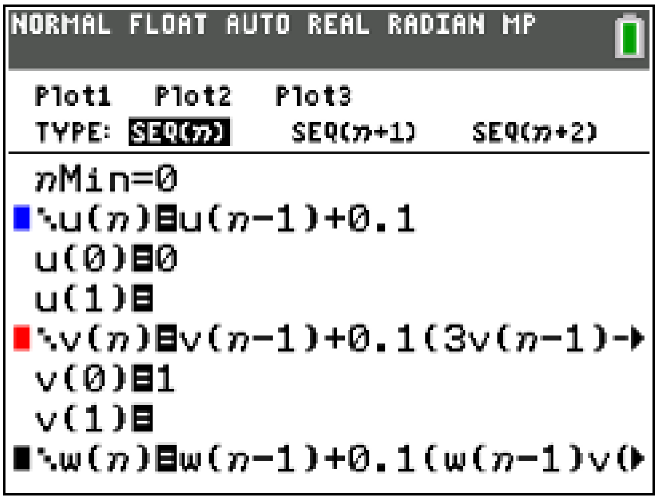
To aid the entry of the sequences the 

button enters *n* and 

allows various options for the entry of the functions.

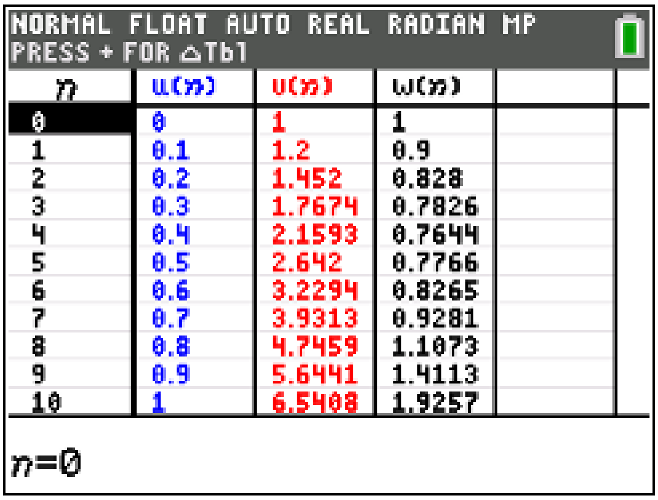


The question above can be answered with the following sequences:



The first sequence entered gives the values of t, the second gives $\dot{x}$ and the third gives $\dot{y}$.

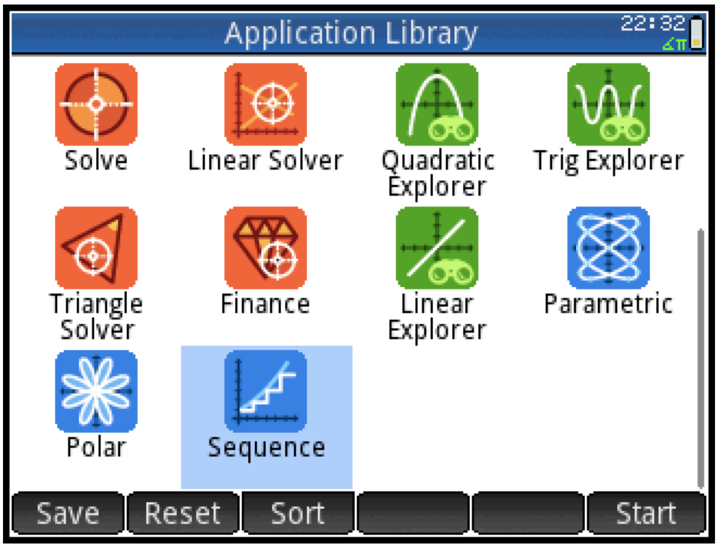
The results can be seen in the table, 2nd table f5 graph

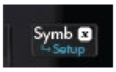


n	u(n)	v(n)	w(n)
0	0	1	1
1	0.1	1.2	0.9
2	0.2	1.452	0.828
3	0.3	1.7674	0.7826
4	0.4	2.1593	0.7644
5	0.5	2.642	0.7766
6	0.6	3.2294	0.8265
7	0.7	3.9313	0.9281
8	0.8	4.7459	1.1073
9	0.9	5.6441	1.4113
10	1	6.5408	1.9257

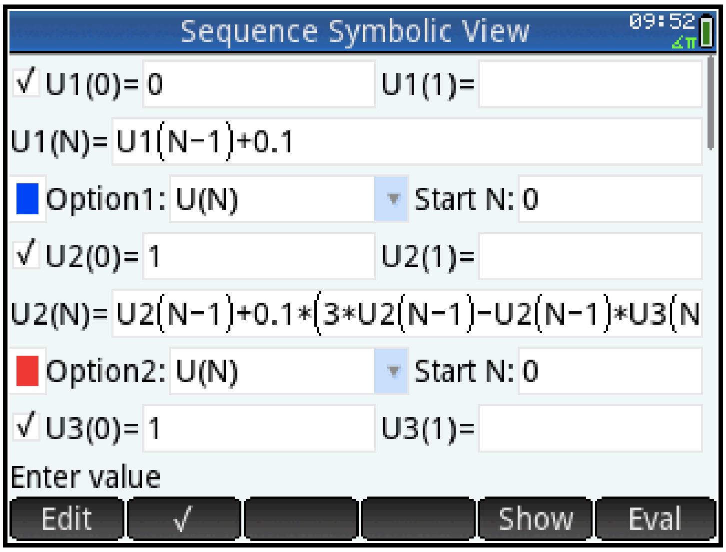
HP-Prime

Sequence is selected from the Apps menu.



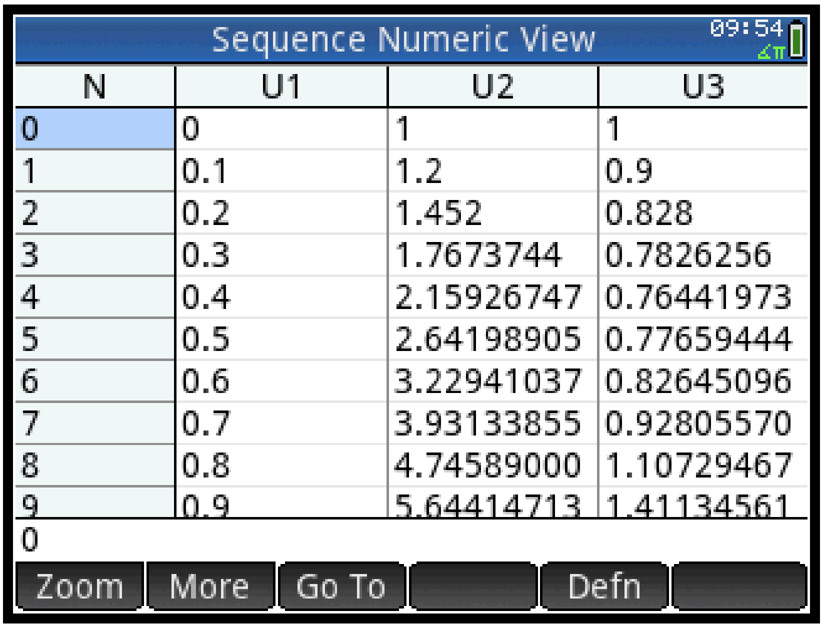
The 

takes you to the function entry page and the sequences are entered.



The 

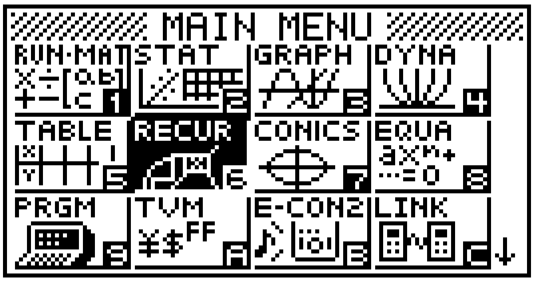
will display the tables of values.



N	U1	U2	U3
0	0	1	1
1	0.1	1.2	0.9
2	0.2	1.452	0.828
3	0.3	1.7673744	0.7826256
4	0.4	2.15926747	0.76441973
5	0.5	2.64198905	0.77659444
6	0.6	3.22941037	0.82645096
7	0.7	3.93133855	0.92805570
8	0.8	4.74589000	1.10729467
9	0.9	5.64414713	1.41134561

Casio fx9750GII

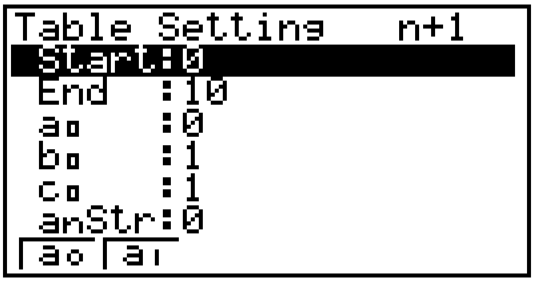
Choose **RECUR** from the apps menu.



To enter the sequences use  or **F4** which gives the options for the terms as below.

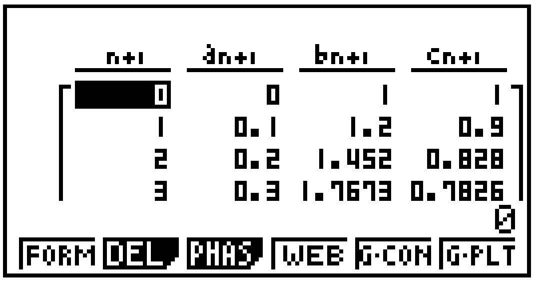
Enter the functions.

Select SET to enter the initial conditions.



Exit again to the screen below.

And then choose TABL to display the table of results.



Contexts

Common contexts for these questions include predator-prey models, population changes and the spread of diseases (for example using the SIR model).

Second-order differential equations

Within the course these are viewed as an extension of the methods of AHL5.16 and AHL5.17. Direct approaches using, for example, an "auxiliary equation" will not be expected in exams, even when this approach is possible.

This method of writing second and higher-order differential equations as a system of linear equations is a frequently-used technique.

The questions in an exam will often be set in a context but knowledge of the context from outside the syllabus will not be required. Any interpretation required will be on the general properties of differential equations or will use information given in the question.

Example question

A mass, M, is attached to the end of a spring. Let *x* be the displacement of M (measured in cm) from an equilibrium position and at t=0, let M be at rest with x=2.

The subsequent motion of M can be described by the second order differential equation:

$$\ddot{x} + \dot{x} + 4.25x = 0.$$

a. Write this as a system of two coupled differential equations.

b. (i) Find the eigenvalues of this system.

(ii) Hence sketch the trajectory of M on a phase portrait.

(iii) Indicate on your diagram the first point for $t > 0$ at which the velocity is equal to zero.

(iv) Describe the long-term state of M.

Use Euler's method with a step length of 0.1 to find:

c. (i) The time, to the nearest tenth of a second, at which M again has a maximum (positive) displacement.

(ii) The displacement at this point.

(iii) Indicate this point on the diagram drawn in part (b).

Solution

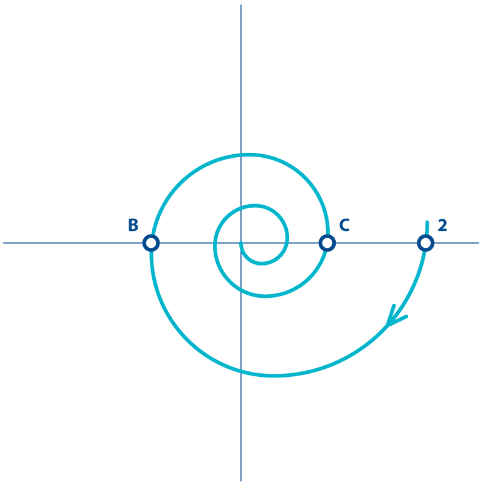
a. $\dot{x} = y$

$\dot{y} = -y - 4.25x.$

$\dot{x}$ = y is defined in the syllabus

$\lambda^2 + \lambda + 4.25 = 0 \Rightarrow \lambda = -0.5 \pm 2i$.

(ii) The trajectory will spiral towards $(0,0)$. At the point $(2,0)$ $\dot{y} = -8.5$ so the spiral is clockwise.



(iii) Indicated as B on the diagram.

(iv) M will come to rest in the equilibrium position.

c. (i) $t = 3.1$.

(ii) $x = 0.8195...$

(iii) Indicated as C on the diagram.