DP Analysis & Approaches HL

Maths with Lemon

Proof by Mathematical Induction

Fully Worked Problems – AHL 1.15

Warm Up: Consecutive Odd Numbers

Question. Does the sum of consecutive odd numbers starting at 1 always equal a perfect square? Prove your claim.

The first n odd numbers are $1, 3, 5, \ldots, 2n-1$. We claim that

$$1 + 3 + \cdots + (2n-1) = n^2.$$

Step 1: Prove the statement is true for $n = 1$

For $n = 1$, LHS $= 1$ and RHS $= 1^2 = 1$. Hence the claim is true.

Step 2: Assume the statement is true for $n = k$

Assume $1 + 3 + \cdots + (2k-1) = k^2$.

Step 3: Prove the statement is true for $n = k + 1$

Add the next odd number, $2k+1$, to both sides:

$$1 + 3 + \cdots + (2k-1) + (2k+1) = k^2 + (2k+1) = (k+1)^2.$$

This is precisely the claim for $n = k + 1$.

Step 4: Conclusion

The statement is true for $n = 1$, and if it is true for $n = k$, it is also true for $n = k + 1$. Therefore, by the principle of mathematical induction, it is true for all positive integers n.

Worked Problem 1: Sum of Squares

Prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}, \quad n \geq 1.$$

Step 1: Prove the statement is true for $n = 1$

LHS $= 1^2 = 1$ and RHS $= \frac{1 \cdot 2 \cdot 3}{6} = 1$. Hence LHS = RHS.

Step 2: Assume the statement is true for $n = k$

Assume

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Step 3: Prove the statement is true for $n = k + 1$

We must show that

$$1^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Using the inductive hypothesis,

$$\begin{aligned} 1^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6}. \end{aligned}$$

This is the required RHS for $n = k + 1$.

Step 4: Conclusion

The statement is true for $n = 1$, and if it is true for $n = k$, it is also true for $n = k + 1$. Therefore, by the principle of mathematical induction, it is true for all positive integers n.

Worked Problem 2: Divisibility

Prove that $7^n - 1$ is divisible by 6 for all integers $n \geq 0$.

Step 1: Prove the statement is true for $n = 0$

The question begins at $n = 0$, so the base case must be $n = 0$ (rather than $n = 1$).

$$7^0 - 1 = 1 - 1 = 0 = 6 \cdot 0.$$

Thus it is divisible by 6.

Step 2: Assume the statement is true for $n = k$

Assume $7^k - 1$ is divisible by 6, so for some integer m,

$$7^k - 1 = 6m \quad \Longrightarrow \quad 7^k = 6m + 1.$$

Step 3: Prove the statement is true for $n = k + 1$

$$\begin{aligned} 7^{k+1} - 1 &= 7 \cdot 7^k - 1 \\ &= 7(6m + 1) - 1 \\ &= 42m + 6 \\ &= 6(7m + 1). \end{aligned}$$

Since $7m + 1$ is an integer, $7^{k+1} - 1$ is divisible by 6.

Step 4: Conclusion

The claim holds for $n = 0$, and the inductive step shows it holds at $k + 1$ whenever it holds at k. Therefore $6 \mid (7^n - 1)$ for every integer $n \geq 0$.

Worked Problem 3: Inequalities

Prove that $8^n \geq 5n$ for all positive integers n.

Step 1: Prove the statement is true for $n = 1$

LHS $= 8^1 = 8$ and RHS $= 5(1) = 5$. Since $8 \geq 5$, the statement holds.

Step 2: Assume the statement is true for $n = k$

Assume $8^k \geq 5k$ for some positive integer k.

Step 3: Prove the statement is true for $n = k + 1$

Multiplying by the positive number 8 gives

$$8^{k+1} = 8 \cdot 8^k \geq 8 \cdot 5k = 40k.$$

For $k \geq 1$,

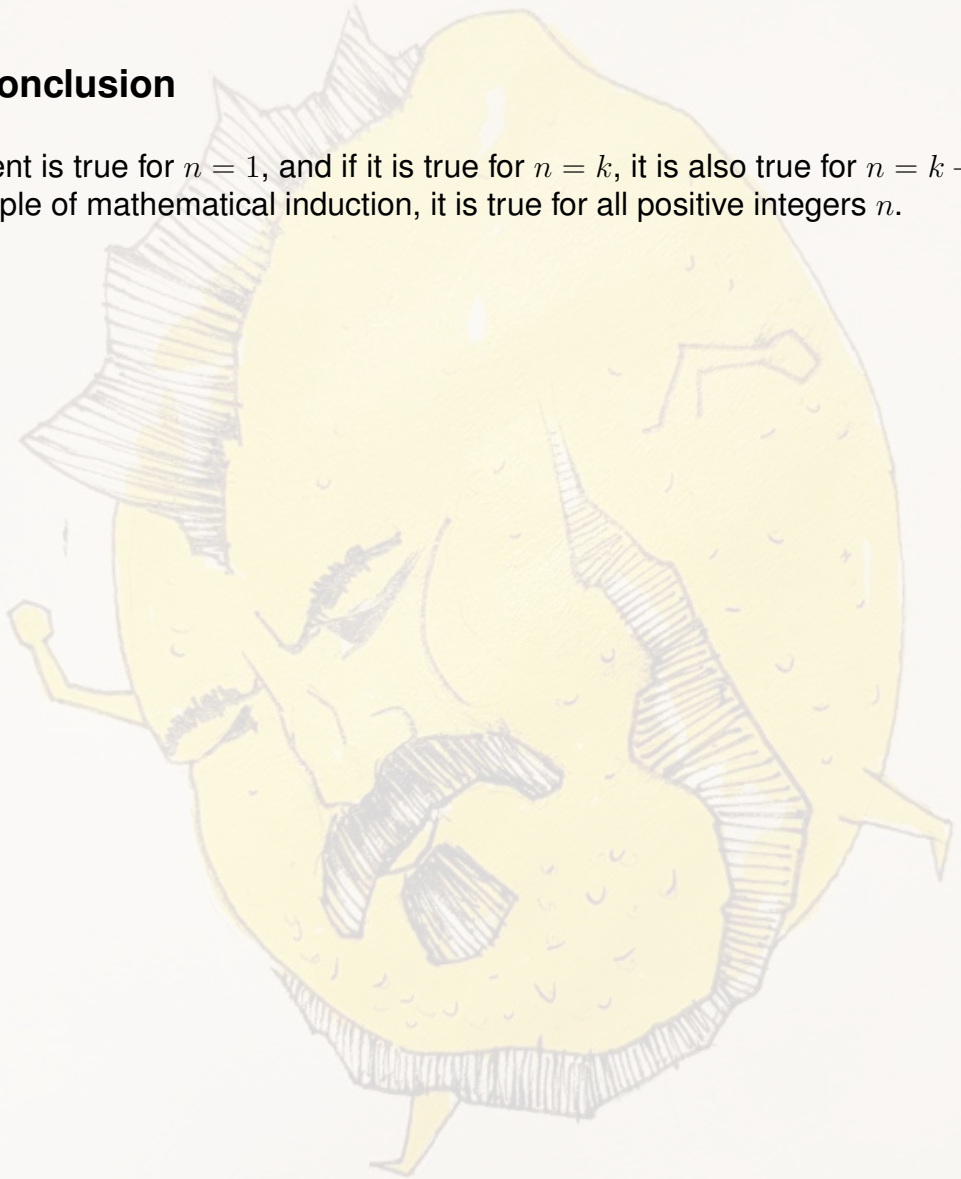
$$40k - 5(k+1) = 35k - 5 \geq 30 > 0,$$

so $40k \geq 5(k+1)$. Therefore

$$\boxed{8^{k+1} \geq 5(k+1)}.$$

Step 4: Conclusion

The statement is true for $n = 1$, and if it is true for $n = k$, it is also true for $n = k + 1$. Therefore, by the principle of mathematical induction, it is true for all positive integers n.



Extra Problem 1: Sum of Cubes

Prove that $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

Step 1: Prove the statement is true for $n = 1$

LHS $= 1$; RHS $= \frac{1^2(2)^2}{4} = 1$.

Step 2: Assume the statement is true for $n = k$

Assume $\sum_{r=1}^{k} r^3 = \frac{k^2(k+1)^2}{4}$.

Step 3: Prove the statement is true for $n = k + 1$

$$\begin{aligned}\sum_{r=1}^{k+1} r^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} \\ &= \frac{(k+1)^2(k^2+4k+4)}{4} \\ &= \boxed{\frac{(k+1)^2(k+2)^2}{4}}.\end{aligned}$$

Step 4: Conclusion

The statement is true for $n = 1$, and if it is true for $n = k$, it is also true for $n = k + 1$. Therefore, by the principle of mathematical induction, it is true for all positive integers n.

Extra Problem 2: Products in a Sum

Prove that

$$1 \cdot 3 + 2 \cdot 4 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

Step 1: Prove the statement is true for $n = 1$

LHS $= 3$; RHS $= \frac{1 \cdot 2 \cdot 9}{6} = 3$.

Step 2: Assume the statement is true for $n = k$

Assume $\sum_{r=1}^{k} r(r+2) = \frac{k(k+1)(2k+7)}{6}$.

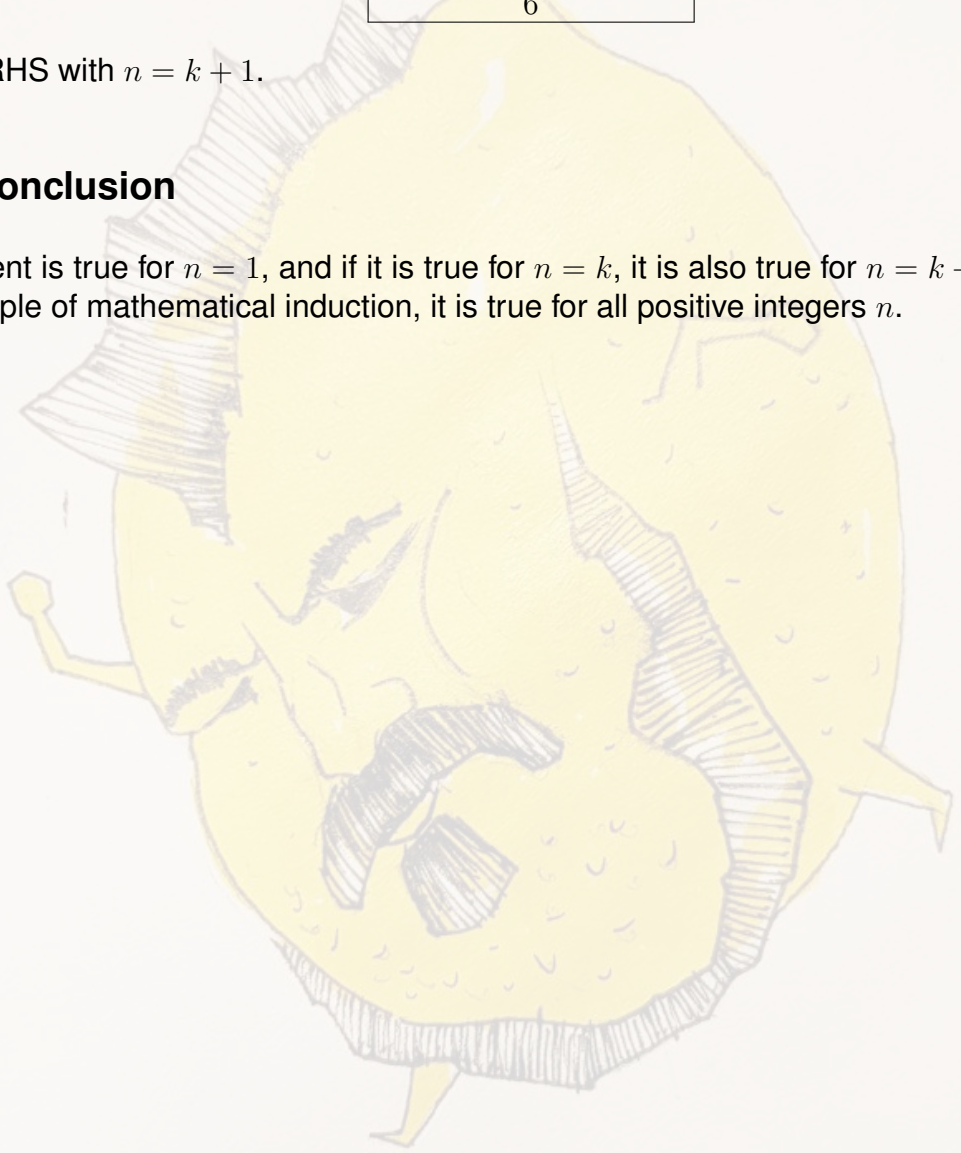
Step 3: Prove the statement is true for $n = k + 1$

$$\begin{aligned}\sum_{r=1}^{k+1} r(r+2) &= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) \\ &= \frac{(k+1)[k(2k+7)+6(k+3)]}{6} \\ &= \frac{(k+1)(2k^2+13k+18)}{6} \\ &= \boxed{\frac{(k+1)(k+2)(2k+9)}{6}}.\end{aligned}$$

This is the RHS with $n = k + 1$.

Step 4: Conclusion

The statement is true for $n = 1$, and if it is true for $n = k$, it is also true for $n = k + 1$. Therefore, by the principle of mathematical induction, it is true for all positive integers n.



Extra Problem 3: Sigma Notation

Prove that

$$\sum_{r=1}^{n} r^2(r+1) = \frac{n(n+1)(n+2)(3n+1)}{12}.$$

Step 1: Prove the statement is true for $n = 1$

LHS $= 1^2(2) = 2$; RHS $= \frac{1 \cdot 2 \cdot 3 \cdot 4}{12} = 2$.

Step 2: Assume the statement is true for $n = k$

Assume

$$\sum_{r=1}^{k} r^2(r+1) = \frac{k(k+1)(k+2)(3k+1)}{12}.$$

Step 3: Prove the statement is true for $n = k + 1$

Add the next term $(k+1)^2(k+2)$:

$$\begin{aligned}
\sum_{r=1}^{k+1} r^2(r+1) &= \frac{k(k+1)(k+2)(3k+1)}{12} + (k+1)^2(k+2) \\
&= \frac{(k+1)(k+2)[k(3k+1) + 12(k+1)]}{12} \\
&= \frac{(k+1)(k+2)(3k^2+13k+12)}{12} \\
&= \boxed{\frac{(k+1)(k+2)(k+3)(3k+4)}{12}}.
\end{aligned}$$

This is the required expression with $n = k + 1$.

Step 4: Conclusion

The statement is true for $n = 1$, and if it is true for $n = k$, it is also true for $n = k + 1$. Therefore, by the principle of mathematical induction, it is true for all positive integers n.

Extra Problem 4: Geometric Sequence

Given $u_n = 2 \cdot 3^{n-1}$, prove that

$$u_1 + u_2 + \cdots + u_n = 3^n - 1.$$

Step 1: Prove the statement is true for $n = 1$

LHS $= u_1 = 2 \cdot 3^0 = 2$; RHS $= 3^1 - 1 = 2$.

Step 2: Assume the statement is true for $n = k$

Assume $u_1 + \cdots + u_k = 3^k - 1$.

Step 3: Prove the statement is true for $n = k + 1$

Since $u_{k+1} = 2 \cdot 3^k$,

$$\begin{aligned} u_1 + \cdots + u_k + u_{k+1} &= (3^k - 1) + 2 \cdot 3^k \\ &= 3 \cdot 3^k - 1 \\ &= \boxed{3^{k+1} - 1}. \end{aligned}$$

Step 4: Conclusion

The statement is true for $n = 1$, and if it is true for $n = k$, it is also true for $n = k + 1$. Therefore, by the principle of mathematical induction, it is true for all positive integers n.



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Mathematics made clear.