

DP Analysis & Approaches (SL)

Maths with Lemon

Lesson 2.8: Function Inverses — Theory Notes

1. What is an Inverse Function?

An **inverse function** reverses, or “undoes”, the effect of another function.

For example, consider

$$f(x) = x + 5.$$

The function adds 5 to the input. Its inverse must therefore subtract 5:

$$f^{-1}(x) = x - 5.$$

For example,

$$3 \xrightarrow{f} 8$$

and the inverse reverses this:

$$8 \xrightarrow{f^{-1}} 3.$$

Therefore,

$$f^{-1}(8) = 3.$$

Key Idea

If

$$f(a) = b,$$

then

$$f^{-1}(b) = a.$$

The notation

$$f^{-1}(x)$$

means the **inverse function** of f .

It does **not** mean

$$\frac{1}{f(x)}.$$

In general,

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

2. Inverse Functions and Composition

If two functions are inverses, applying one function followed by the other returns the original input.

Therefore,

$$(f \circ f^{-1})(x) = x$$

and

$$(f^{-1} \circ f)(x) = x.$$

The function

$$I(x) = x$$

is called the **identity function**.

Therefore,

$$f \circ f^{-1} = I$$

and

$$f^{-1} \circ f = I.$$

Example

Let

$$f(x) = x + 2$$

and

$$g(x) = x - 2.$$

Then

$$(f \circ g)(x) = f(x - 2)$$

$$= (x - 2) + 2$$

$$= x.$$

Similarly,

$$(g \circ f)(x) = g(x + 2)$$

$$= (x + 2) - 2$$

$$= x.$$

Therefore,

$$f \text{ and } g \text{ are inverse functions.}$$

3. How to Check Whether Two Functions are Inverses

Suppose we are given two functions f and g .

To show that they are inverses, calculate

$$(f \circ g)(x)$$

and

$$(g \circ f)(x).$$

If both simplify to x , then

$$\boxed{g = f^{-1}}.$$

Example

Consider

$$f(x) = 2x - 2$$

and

$$g(x) = \frac{x}{2} + 1.$$

First,

$$(f \circ g)(x) = 2\left(\frac{x}{2} + 1\right) - 2.$$

Therefore,

$$(f \circ g)(x) = x + 2 - 2 = x.$$

Now,

$$(g \circ f)(x) = \frac{2x - 2}{2} + 1.$$

Thus,

$$(g \circ f)(x) = x - 1 + 1 = x.$$

Hence,

$$\boxed{f^{-1}(x) = g(x)}.$$

4. Finding an Inverse Algebraically

To find the inverse of a function:

1. Write

$$y = f(x).$$

2. Swap x and y .

3. Rearrange the equation to make y the subject.

4. Replace y with

$$f^{-1}(x).$$

Key Idea

To find an inverse:

$$\boxed{y = f(x) \rightarrow \text{swap } x \text{ and } y \rightarrow \text{solve for } y}$$

5. Inverse of a Linear Function

Consider

$$f(x) = 2x + 6.$$

Start with

$$y = 2x + 6.$$

Swap x and y :

$$x = 2y + 6.$$

Solve for y :

$$x - 6 = 2y,$$

so

$$y = \frac{x - 6}{2}.$$

Therefore,

$$f^{-1}(x) = \frac{x - 6}{2}.$$

General Result

Suppose

$$f(x) = ax + b, \quad a \neq 0.$$

Then

$$y = ax + b.$$

Swap:

$$x = ay + b.$$

Hence

$$x - b = ay$$

and

$$y = \frac{x - b}{a}.$$

Therefore,

$$f^{-1}(x) = \frac{x - b}{a}.$$

6. Example with Fractions

Suppose

$$f(x) = -\frac{4}{5}x - 4.$$

Write

$$y = -\frac{4}{5}x - 4.$$

Swap x and y :

$$x = -\frac{4}{5}y - 4.$$

Add 4:

$$x + 4 = -\frac{4}{5}y.$$

Multiply by $-\frac{5}{4}$:

$$y = -\frac{5}{4}(x + 4).$$

Therefore,

$$\boxed{f^{-1}(x) = -\frac{5}{4}(x + 4)}.$$

7. Inverse of a Radical Function

The same process works for functions containing roots and powers.

Consider

$$h(x) = \sqrt[3]{\frac{x-1}{2}}.$$

Write

$$y = \sqrt[3]{\frac{x-1}{2}}.$$

Cube both sides:

$$y^3 = \frac{x-1}{2}.$$

Multiply by 2:

$$2y^3 = x - 1.$$

Therefore,

$$x = 2y^3 + 1.$$

Swap the variables:

$$\boxed{h^{-1}(x) = 2x^3 + 1}.$$

Checking the Result

$$h^{-1}(h(x)) = 2 \left(\sqrt[3]{\frac{x-1}{2}} \right)^3 + 1.$$

Therefore,

$$h^{-1}(h(x)) = 2 \left(\frac{x-1}{2} \right) + 1 = x.$$

8. Another Radical Example

Suppose

$$f(x) = (x^5 + 2)^{1/4}.$$

Write

$$y = (x^5 + 2)^{1/4}.$$

Raise both sides to the fourth power:

$$y^4 = x^5 + 2.$$

Therefore,

$$x^5 = y^4 - 2.$$

Take the fifth root:

$$x = \sqrt[5]{y^4 - 2}.$$

Hence,

$$\boxed{f^{-1}(x) = \sqrt[5]{x^4 - 2}}.$$

9. Inverse of a Rational Function

Consider

$$g(x) = \frac{x-3}{x+2}.$$

Write

$$y = \frac{x-3}{x+2}.$$

Multiply both sides by $x+2$:

$$y(x+2) = x-3.$$

Expand:

$$yx + 2y = x - 3.$$

Collect the terms containing x :

$$yx - x = -3 - 2y.$$

Factor:

$$x(y-1) = -(3+2y).$$

Therefore,

$$x = \frac{3+2y}{1-y}.$$

Replace y with x :

$$g^{-1}(x) = \frac{2x + 3}{1 - x}.$$

10. Domain and Range of Inverse Functions

The domain and range exchange roles when we find an inverse.

If

$$f : A \rightarrow B,$$

then

$$f^{-1} : B \rightarrow A.$$

Therefore,

$$\text{Domain}(f^{-1}) = \text{Range}(f)$$

and

$$\text{Range}(f^{-1}) = \text{Domain}(f).$$

For example, suppose

$$f(5) = 1.$$

Then

$$f^{-1}(1) = 5.$$

The input and output have exchanged roles.

11. One-to-One Functions

A function can only have an inverse function if it is **one-to-one** on its domain.

A function is one-to-one when different inputs produce different outputs.

In other words,

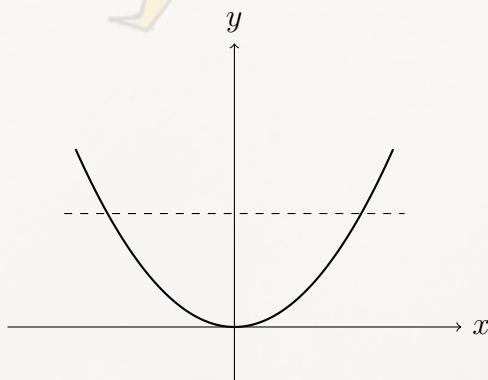
$$f(a) = f(b)$$

implies

$$a = b.$$

Horizontal Line Test

A graph represents a one-to-one function if every horizontal line intersects the graph at most once.



A quadratic fails the horizontal line test.

For example,

$$f(x) = x^2$$

is not one-to-one over all real numbers because

$$f(2) = 4$$

and

$$f(-2) = 4.$$

12. Restricting the Domain

Sometimes we can make a function one-to-one by restricting its domain. Consider

$$f(x) = x^2.$$

If we restrict the domain to

$$x \geq 0,$$

then f becomes one-to-one.
Its inverse is

$$f^{-1}(x) = \sqrt{x}.$$

Without the domain restriction,

$$\sqrt{x^2} = |x|,$$

not necessarily x .

Therefore, domain restrictions are particularly important when working with quadratic functions and square roots.

13. Graphs of Inverse Functions

The graph of an inverse function is the reflection of the original function in the line

$$y = x.$$

If a point

$$(a, b)$$

lies on f , then

$$(b, a)$$

lies on f^{-1} .

For example,

$$(-4, -2)$$

becomes

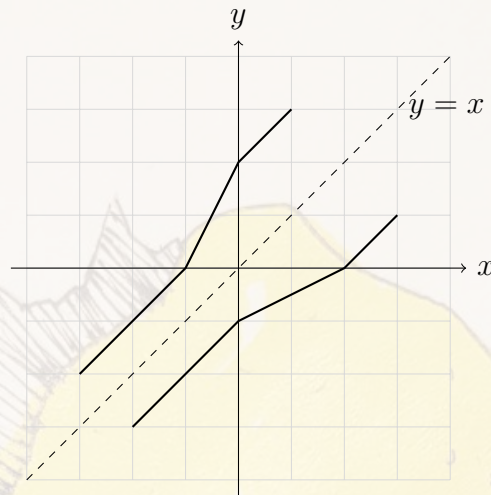
$$(-2, -4).$$

Similarly,

$(0, 3)$

becomes

$(3, 0)$.



Key Idea

Finding an inverse algebraically means swapping x and y .
Finding an inverse graphically also swaps the x - and y -coordinates.
These are the same idea.

14. Finding Inverses from Tables

Suppose

x	0	1	2	3	4	5	6
$f(x)$	3	6	4	2	0	1	5

From the table,

$$f(1) = 6.$$

Therefore,

$$f^{-1}(6) = 1.$$

Also,

$$f(5) = 1.$$

Therefore,

$$f^{-1}(1) = 5.$$

To find $f^{-1}(a)$ from a table, ask:

Which input gives the output a ?

15. Evaluating an Inverse Without Finding its Formula

Suppose

$$f(x) = \frac{x}{2} + 5$$

and we want to find

$$f^{-1}(7).$$

This means:

What value of x gives $f(x) = 7$?

Therefore solve

$$\frac{x}{2} + 5 = 7.$$

Then

$$\frac{x}{2} = 2$$

and

$$x = 4.$$

Hence,

$$\boxed{f^{-1}(7) = 4}.$$

Key Idea

Finding $f^{-1}(a)$ is equivalent to solving

$$\boxed{f(x) = a}.$$

16. Composition with Inverse Functions

Inverse functions can appear inside compositions.

For example,

$$f^{-1}(g^{-1}(x)).$$

Always work from the **inside out**.

If

$$g(x) = x + 7,$$

then

$$g^{-1}(x) = x - 7.$$

If an equation gives

$$f^{-1}(g^{-1}(x)) = 9,$$

we can apply f to both sides:

$$g^{-1}(x) = f(9).$$

This often avoids having to calculate a complicated formula for f^{-1} explicitly.

17. Important Interpretation

Suppose

$$f(4) = 11.$$

This immediately tells us

$$\boxed{f^{-1}(11) = 4}.$$

Similarly, if

$$f^{-1}(7) = 3,$$

then

$$\boxed{f(3) = 7}.$$

The inverse simply reverses the relationship between input and output.

18. Common Mistakes

- Confusing

with

$$f^{-1}(x)$$

$$\frac{1}{f(x)}.$$

- Forgetting to swap x and y when finding an inverse.
- Forgetting that the domain and range exchange roles.
- Assuming every function has an inverse.
- Forgetting domain restrictions for quadratic and other non-one-to-one functions.
- Writing

$$\sqrt{x^2} = x$$

without considering the domain. In general,

$$\sqrt{x^2} = |x|.$$

- Reflecting an inverse graph in the x -axis instead of in

$$y = x.$$

19. Formula and Concept Summary

Concept	Rule
Inverse notation	$f^{-1}(x)$
Inverse relationship	$f(a) = b \iff f^{-1}(b) = a$
Composition	$f(f^{-1}(x)) = x$
Reverse composition	$f^{-1}(f(x)) = x$
Domain/range	$D_{f^{-1}} = R_f, \quad R_{f^{-1}} = D_f$
Graph	Reflection in $y = x$
Point transformation	$(a, b) \longrightarrow (b, a)$
Linear inverse	If $f(x) = ax + b,$ then $f^{-1}(x) = \frac{x - b}{a}$
Existence of inverse	f must be one-to-one on its domain.

20. Final Checklist

Before finishing an inverse-function question, ask:

1. Is the function one-to-one?
2. Do I need to restrict the domain?
3. Have I swapped x and y correctly?
4. Have I solved completely for the new y ?
5. Does

$$f(f^{-1}(x)) = x$$

hold?

6. Have the domain and range exchanged?

7. If I am sketching, have I reflected the graph in

$$y = x?$$





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Mathematics made clear.