DP Analysis & Approaches (SL/HL)

Maths with Lemon

Lesson 3.5 Rational Functions

Mathematician: _____

Syllabus	Content and guidance
SL 2.8	$f(x) = \frac{ax+b}{cx+d}$, vertical and horizontal asymptotes, intercepts, sketches.
AHL 2.13	Rational functions with quadratic denominators and linear denominators; horizontal, vertical and oblique asymptotes.

Warm Up

Let $f(x) = 16 - x^2$.

(a) Find the x-intercepts. [2]

(b) A rectangle is inscribed symmetrically under the parabola, with half-width $a > 0$. Show its area is $A = 32a - 2a^3$. [2]

(c) Find $a > 0$ maximizing the area. [3]

Let $g(x) = (x-4)^2 + k$.

(d) Show that intersections satisfy $2x^2 - 8x + k = 0$. [4]

(e) If the graphs intersect exactly once, find k. [5]

Points of Discontinuity

A rational function is undefined wherever its denominator is zero. State the excluded values:

(a) $\dfrac{x^2 - 36}{x - 8}$

(b) $\dfrac{x^2 - 4}{8x + 5}$

(c) $\dfrac{7x + 56}{x^2 - 36}$

Rational Functions: Key Features

A rational function has the form $r(x) = p(x)/q(x)$, where p, q are polynomials and $q(x) \neq 0$.

- Vertical asymptotes occur at uncancelled denominator zeros.
- Removable discontinuities (holes) occur at cancelled factors.
- x-intercepts occur at zeros of the simplified numerator that remain in the domain.
- The y-intercept is $f(0)$, if defined.
- If numerator degree is smaller than denominator degree, the horizontal asymptote is $y = 0$.
- If degrees are equal, the horizontal asymptote is the ratio of leading coefficients.
- For $\frac{ax+b}{cx+d}$, $x = -d/c$ and $y = a/c$, provided $ad - bc \neq 0$.

Horizontal Asymptotes: Guided Examples

Worked Example	You Try
Find the asymptotes $\frac{1}{x}$ $x = 0, \quad y = 0.$ The numerator degree is smaller.	**Find the asymptotes** $\frac{1}{2x-8}$ *Show your working.* . . . .

Worked Example	You Try
Find the asymptotes $\frac{3x+1}{x-5}$ $x = 5, \quad y = 3.$ Equal degrees: leading-coefficient ratio 3/1.	**Find the asymptotes** $\frac{x+1}{2x-5}$ *Show your working.* . . . .

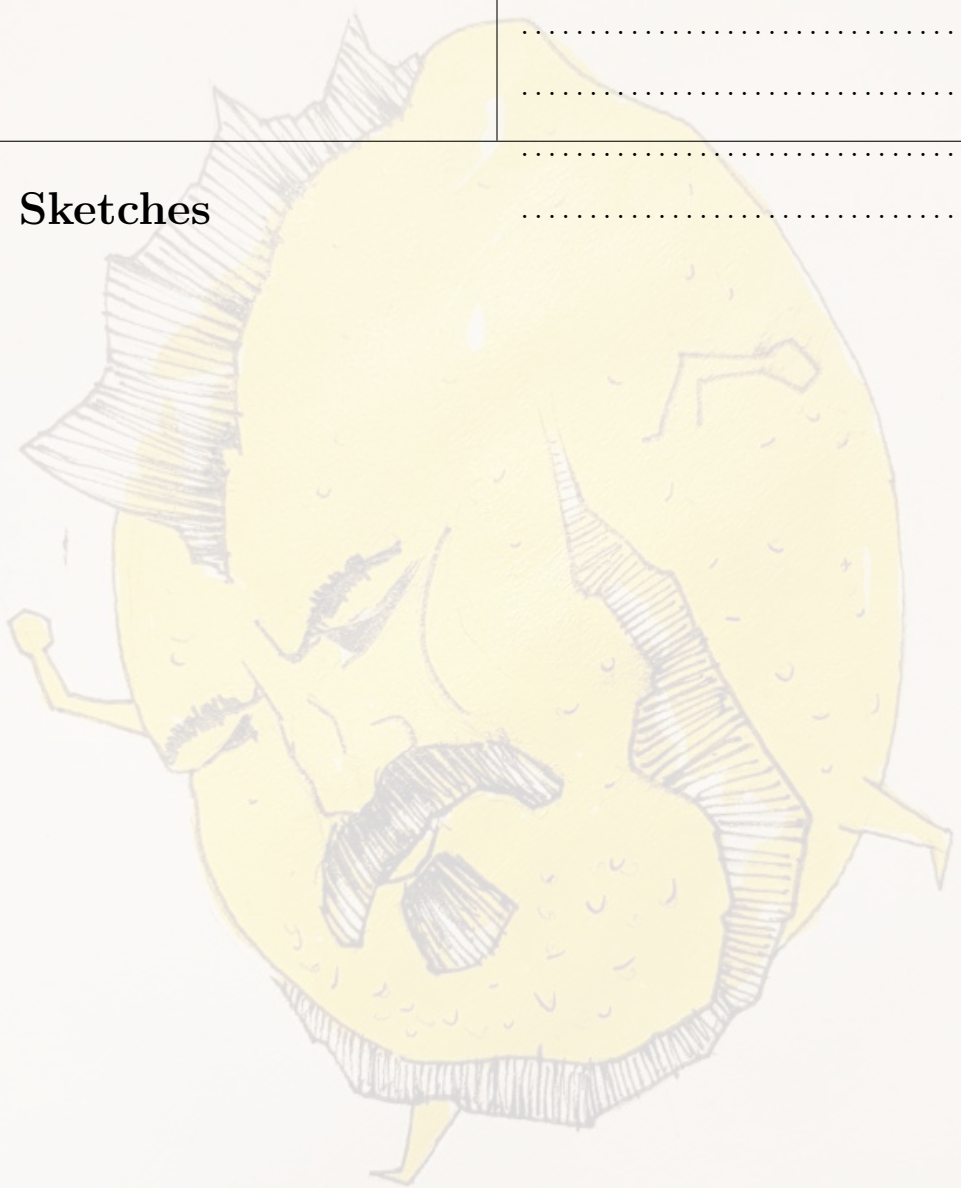
Worked Example	You Try
Describe the transformation $\frac{1}{x} + 4$ The graph of $y = 1/x$ is translated **up 4 units**. The horizontal asymptote becomes $y = 4$.	**Describe the transformation** $\frac{1}{x-3}$ *Show your working.* . . . .

Sketching Rational Functions

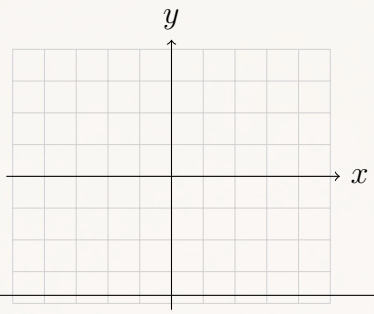
For each graph, label vertical/horizontal asymptotes, intercepts and branch positions.

Worked Example	You Try
Sketch and label $\dfrac{2x+3}{x+4}$ $x=-4,\ y=2.$ x-intercept $(-3/2, 0)$; y-intercept $(0, 3/4)$. $y = 2 - \dfrac{5}{x+4}$.	**Sketch and label** $\dfrac{2x+3}{x+1}$ *Show your working.* . . . .

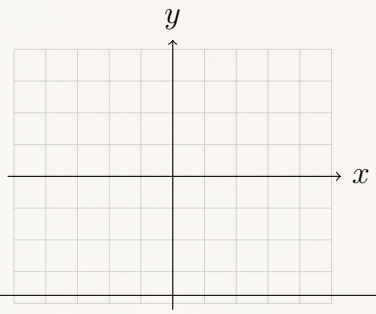
You Try: Sketches



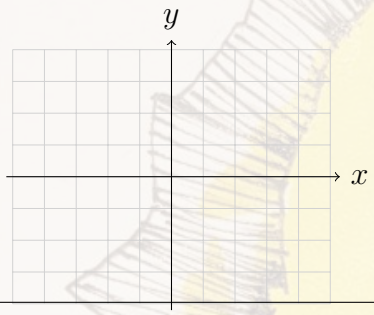
$y = \dfrac{2x+3}{x+1}$



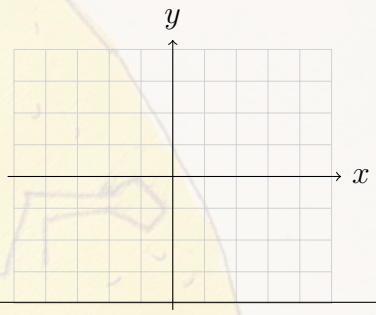
$y = \dfrac{3x+1}{x+2}$



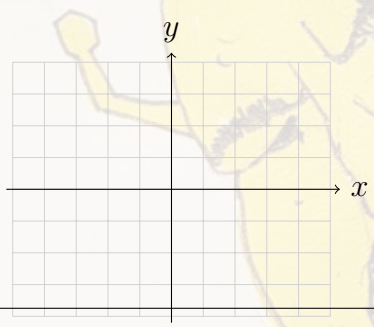
$y = \dfrac{x}{x-5}$



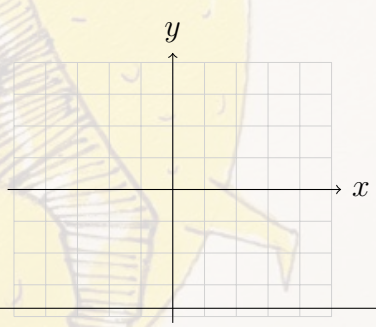
$y = \dfrac{x}{x+3}$



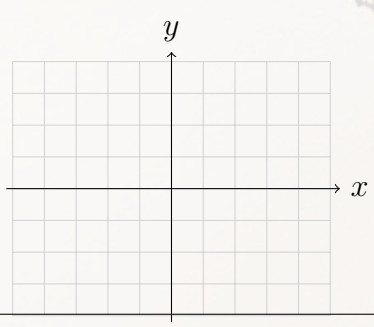
$y = \dfrac{3}{2x+1}$



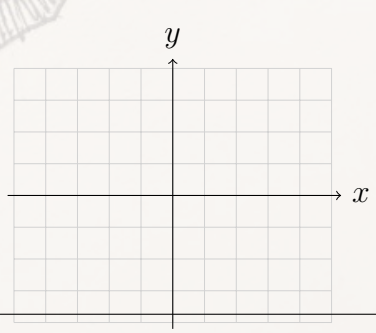
$y = \dfrac{2}{3x-1}$



$y = \dfrac{x}{x-5} + 3$



$y = \dfrac{2}{3x-1} - 4$



Describing Transformations

Describe transformations from $y = 1/x$.

(a) $y = \dfrac{1}{x} + 2$

(b) $y = \dfrac{1}{x-3}$

(c) $y = \dfrac{1}{x+2}$

(d) $y = -\dfrac{2}{x-3}$

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Challenge Problems

Problem 1. For $y = \frac{2x+5}{x}$: (a) write as $a + b/x$; (b) describe two transformations from $1/x$; (c) state asymptotes.

Problem 1

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Problem 2. For $y = \frac{2x-5}{x-3}$: (a) show $2 + \frac{1}{x-3}$; (b) describe two transformations; (c) sketch with asymptotes.

Problem 2

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Problem 3. For $y = \frac{5x-1}{x}$: (a) write as $a - b/x$; (b) describe two transformations; (c) sketch, giving intercepts and asymptotes.

Problem 3

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Problem 4. (a) Simplify $\frac{x^2+2x}{x^2-4}$. (b) Sketch, showing all discontinuities.

Problem 4

Problem 5. Sketch $y = \frac{x-a}{x-b}$ for (a) $0 < a < b$ and (b) $0 < b < a$. Label asymptotes and intercepts.

Problem 5



Detailed Markscheme

Lesson 3.5 Rational Functions

Warm Up [16 marks]

(a) [**2**] $16 - x^2 = 0 \Rightarrow x^2 = 16 \Rightarrow \boxed{x = \pm 4}$. (M1 A1)

(b) [**2**] Rectangle width $2a$, height $16 - a^2$.

$$A = 2a(16 - a^2) = \boxed{32a - 2a^3}.$$

(M1 A1)

(c) [**3**] Differentiate:

$$A'(a) = 32 - 6a^2 = 0 \Rightarrow a^2 = \frac{16}{3} \Rightarrow \boxed{a = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}}.$$

Since $A''(a) = -12a < 0$ for $a > 0$, it is a maximum. (M1 A1 R1)

(d) [**4**] Set $16 - x^2 = (x-4)^2 + k = x^2 - 8x + 16 + k$. Hence

$$16 - x^2 = x^2 - 8x + 16 + k \Rightarrow \boxed{2x^2 - 8x + k = 0}.$$

(M1 M1 A1 A1)

(e) [**5**] One intersection requires discriminant zero:

$$(-8)^2 - 4(2)k = 0 \Rightarrow 64 - 8k = 0 \Rightarrow \boxed{k = 8}.$$

(M1 M1 A1 R1 A1)

Points of Discontinuity [3 marks]

(a) $x - 8 = 0 \Rightarrow \boxed{x = 8}$.

(b) $8x + 5 = 0 \Rightarrow \boxed{x = -5/8}$.

(c) $x^2 - 36 = (x-6)(x+6) = 0 \Rightarrow \boxed{x = \pm 6}$.

Award A1 each.

Guided Examples: You Try [6 marks]

1. $\frac{1}{2x-8}$: vertical $x = 4$, horizontal $y = 0$.
2. $\frac{x+1}{2x-5}$: vertical $x = 5/2$, horizontal $y = 1/2$.
3. $\frac{1}{x-3}$: translation right 3 units; asymptotes $x = 3, y = 0$.
4. $\frac{2x+3}{x+1} = 2 + \frac{1}{x+1}$: vertical $x = -1$, horizontal $y = 2$; intercepts $(-3/2, 0)$, $(0, 3)$.

Eight Graphs: Complete Analytical Features

For each, award M1 for denominator analysis, A1 for horizontal asymptote, A1 for intercepts, and R1 for correct branch placement/sketch (4 marks each).

Function	Vertical	Horizontal
$\frac{2x+3}{x+1}$	$x=-1$	$y=2$
$\frac{3x+1}{x+2}$	$x=-2$	$y=3$
$\frac{x}{x-5}$	$x=5$	$y=1$
$\frac{x}{x+3}$	$x=-3$	$y=1$
$\frac{3}{2x+1}$	$x=-1/2$	$y=0$
$\frac{2}{3x-1}$	$x=1/3$	$y=0$
$\frac{x}{x-5}+3$	$x=5$	$y=4$
$\frac{2}{3x-1}-4$	$x=1/3$	$y=-4$

For branch orientation, rewrite each function as $y=k+\frac{A}{x-h}$. If $A>0$, branches lie above-right and below-left of (h,k); if $A<0$, they lie above-left and below-right.

Transformations [5 marks]

(a) $\frac{1}{x}+2$: up 2.

(b) $\frac{1}{x-3}$: right 3.

(c) $\frac{1}{x+2}$: left 2.

(d) $-\frac{2}{x-3}$: right 3, reflection in the x-axis, vertical stretch factor 2.

Challenge 1 [5 marks]

$$\frac{2x+5}{x}=\boxed{2+\frac{5}{x}}.$$

From $1/x$, stretch vertically by factor 5 and translate up 2. Asymptotes: $\boxed{x=0,\ y=2}$.

Challenge 2 [6 marks]

$$\frac{2x-5}{x-3}=\frac{2(x-3)+1}{x-3}=\boxed{2+\frac{1}{x-3}}.$$

Translate $1/x$ right 3 and up 2. Asymptotes $x=3, y=2$. Intercepts:

$$2x-5=0\Rightarrow x=\frac{5}{2},\qquad y(0)=\frac{-5}{-3}=\frac{5}{3}.$$

The branch for $x>3$ lies above $y=2$, and for $x<3$ below $y=2$.

Challenge 3 [6 marks]

$$\frac{5x-1}{x}=\boxed{5-\frac{1}{x}}.$$

Reflect $1/x$ in the x-axis, then translate up 5. Asymptotes $x = 0, y = 5$. x-intercept: $5x - 1 = 0 \Rightarrow \boxed{x = 1/5}$. No y-intercept since $x = 0$ is excluded.

Challenge 4 [6 marks]

$$\frac{x^2 + 2x}{x^2 - 4} = \frac{x(x+2)}{(x-2)(x+2)} = \boxed{\frac{x}{x-2}}, \qquad x \neq -2, 2.$$

At $x = -2$, the cancelled factor gives a **hole**, not an asymptote:

$$y = \frac{-2}{-4} = \boxed{\frac{1}{2}}, \quad \text{hole at } (-2, 1/2).$$

Vertical asymptote $x = 2$, horizontal asymptote $y = 1$. The graph passes through $(0, 0)$. Rewrite $y = 1 + \frac{2}{x-2}$ to determine branch orientation.

Challenge 5 [8 marks]

$$\frac{x-a}{x-b} = 1 + \frac{b-a}{x-b}, \qquad a \neq b.$$

In both cases the vertical asymptote is $\boxed{x = b}$, horizontal asymptote $\boxed{y = 1}$, x-intercept $\boxed{(a, 0)}$, and y-intercept $\boxed{(0, a/b)}$.

(a) $0 < a < b$**:** $b - a > 0$. The right branch ($x > b$) is above $y = 1$, and the left branch ($x < b$) is below $y = 1$. Also $0 < a/b < 1$.

(b) $0 < b < a$**:** $b - a < 0$. The right branch ($x > b$) is below $y = 1$, and the left branch ($x < b$) is above $y = 1$. Also $a/b > 1$.



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Mathematics made clear.