

DP Analysis & Approaches SL / HL

Maths with Lemon

Lesson 9.2 Integration by Substitution — Detailed Solutions

Warm Up

a)

$$\int 5x^7 dx = \frac{5}{8}x^8 + C.$$

$$\boxed{\frac{5}{8}x^8 + C}$$

b)

$$\int (8x^{3/2} - 2 + x^{-2}) dx = \frac{16}{5}x^{5/2} - 2x - \frac{1}{x} + C.$$

$$\boxed{\frac{16}{5}x^{5/2} - 2x - \frac{1}{x} + C}$$

c)

$$\int (x^{-3} + x^{-1}) dx = -\frac{1}{2x^2} + \ln|x| + C.$$

$$\boxed{-\frac{1}{2x^2} + \ln|x| + C}$$

Example 1

Let

$$u = 5x + 4, \quad du = 5 dx, \quad dx = \frac{1}{5} du.$$

Then

$$\int (5x + 4)^5 dx = \frac{1}{5} \int u^5 du = \frac{u^6}{30} + C.$$

Therefore

$$\boxed{\frac{(5x + 4)^6}{30} + C}.$$

Example 2

Let

$$u = 3x + 2, \quad du = 3 dx, \quad dx = \frac{1}{3} du.$$

Thus

$$\int (3x + 2)^4 dx = \frac{1}{3} \int u^4 du = \frac{u^5}{15} + C,$$

so

$$\boxed{\frac{(3x + 2)^5}{15} + C}.$$

Example 3

Let

$$u = t^3 + 4, \quad du = 3t^2 dt.$$

Hence

$$\int 3t^2(t^3 + 4)^5 dt = \int u^5 du = \frac{u^6}{6} + C.$$

$$\boxed{\frac{(t^3 + 4)^6}{6} + C}$$

Example 4

Let

$$u = 4x - 5, \quad du = 4 dx, \quad dx = \frac{1}{4} du.$$

Then

$$\int \sqrt{4x - 5} dx = \frac{1}{4} \int u^{1/2} du = \frac{1}{6} u^{3/2} + C.$$

$$\boxed{\frac{1}{6}(4x - 5)^{3/2} + C}$$

Example 5

Let

$$u = t^3 + 4, \quad du = 3t^2 dt.$$

Therefore

$$\int t^2(t^3 + 4)^{-1/2} dt = \frac{1}{3} \int u^{-1/2} du = \frac{2}{3} u^{1/2} + C.$$

$$\boxed{\frac{2}{3} \sqrt{t^3 + 4} + C}$$

Example 6

Let

$$u = 2x + 1, \quad du = 2 dx.$$

Then

$$\int \cos(2x + 1) dx = \frac{1}{2} \int \cos u du = \frac{1}{2} \sin u + C.$$

$$\boxed{\frac{1}{2} \sin(2x + 1) + C}$$

Example 7

Let

$$u = \sin x, \quad du = \cos x \, dx.$$

Then

$$\int \sin^{10} x \cos x \, dx = \int u^{10} \, du = \frac{u^{11}}{11} + C.$$

$$\boxed{\frac{\sin^{11} x}{11} + C}$$

Example 8

Let

$$u = \sqrt{x} - 1.$$

Then

$$du = \frac{1}{2\sqrt{x}} \, dx \quad \Rightarrow \quad \frac{dx}{\sqrt{x}} = 2 \, du.$$

Therefore

$$\int \frac{(\sqrt{x} - 1)^2}{\sqrt{x}} \, dx = 2 \int u^2 \, du = \frac{2}{3} u^3 + C.$$

$$\boxed{\frac{2}{3}(\sqrt{x} - 1)^3 + C}$$

Example 9

Let

$$u = x^3 + x^2, \quad du = (3x^2 + 2x) \, dx.$$

Hence

$$\int \sqrt{x^3 + x^2} (3x^2 + 2x) \, dx = \int u^{1/2} \, du = \frac{2}{3} u^{3/2} + C.$$

$$\boxed{\frac{2}{3}(x^3 + x^2)^{3/2} + C}$$

Example 10

Let

$$u = -\frac{x}{\pi} + \frac{1}{3}.$$

Then

$$du = -\frac{1}{\pi} dx \quad \Rightarrow \quad dx = -\pi du.$$

Thus

$$\int e^{-x/\pi+1/3} dx = -\pi \int e^u du = -\pi e^u + C.$$

$$\boxed{-\pi e^{-x/\pi+1/3} + C}$$

Example 11

Since

$$\frac{dy}{dx} = x \sin(3x^2),$$

we have

$$y = \int x \sin(3x^2) dx.$$

Let

$$u = 3x^2, \quad du = 6x dx.$$

Then

$$y = \frac{1}{6} \int \sin u du = -\frac{1}{6} \cos u + C.$$

Therefore

$$\boxed{y = -\frac{1}{6} \cos(3x^2) + C}.$$

Example 12

First simplify:

$$\frac{e^{2x} - e^{-2x}}{e^x} = e^x - e^{-3x}.$$

Therefore

$$\int (e^x - e^{-3x}) dx = e^x + \frac{1}{3} e^{-3x} + C.$$

$$\boxed{e^x + \frac{1}{3} e^{-3x} + C}$$

Example 13

As written,

$$\sqrt{x^4 + x^2} = \sqrt{x^2(x^2 + 1)} = |x|\sqrt{x^2 + 1}.$$

For $x \geq 0$, this becomes

$$\int x\sqrt{x^2 + 1} \, dx.$$

Let

$$u = x^2 + 1, \quad du = 2x \, dx.$$

Then

$$\frac{1}{2} \int u^{1/2} \, du = \frac{1}{3} u^{3/2} + C.$$

Hence, for $x \geq 0$,

$$\boxed{\frac{1}{3}(x^2 + 1)^{3/2} + C}.$$

For $x < 0$, $|x| = -x$, so the corresponding antiderivative is

$$\boxed{-\frac{1}{3}(x^2 + 1)^{3/2} + C}.$$

Example 14

Rewrite

$$\tan x = \frac{\sin x}{\cos x}.$$

Let

$$u = \cos x, \quad du = -\sin x \, dx.$$

Then

$$\int \tan x \, dx = - \int \frac{1}{u} \, du = -\ln |u| + C.$$

Therefore

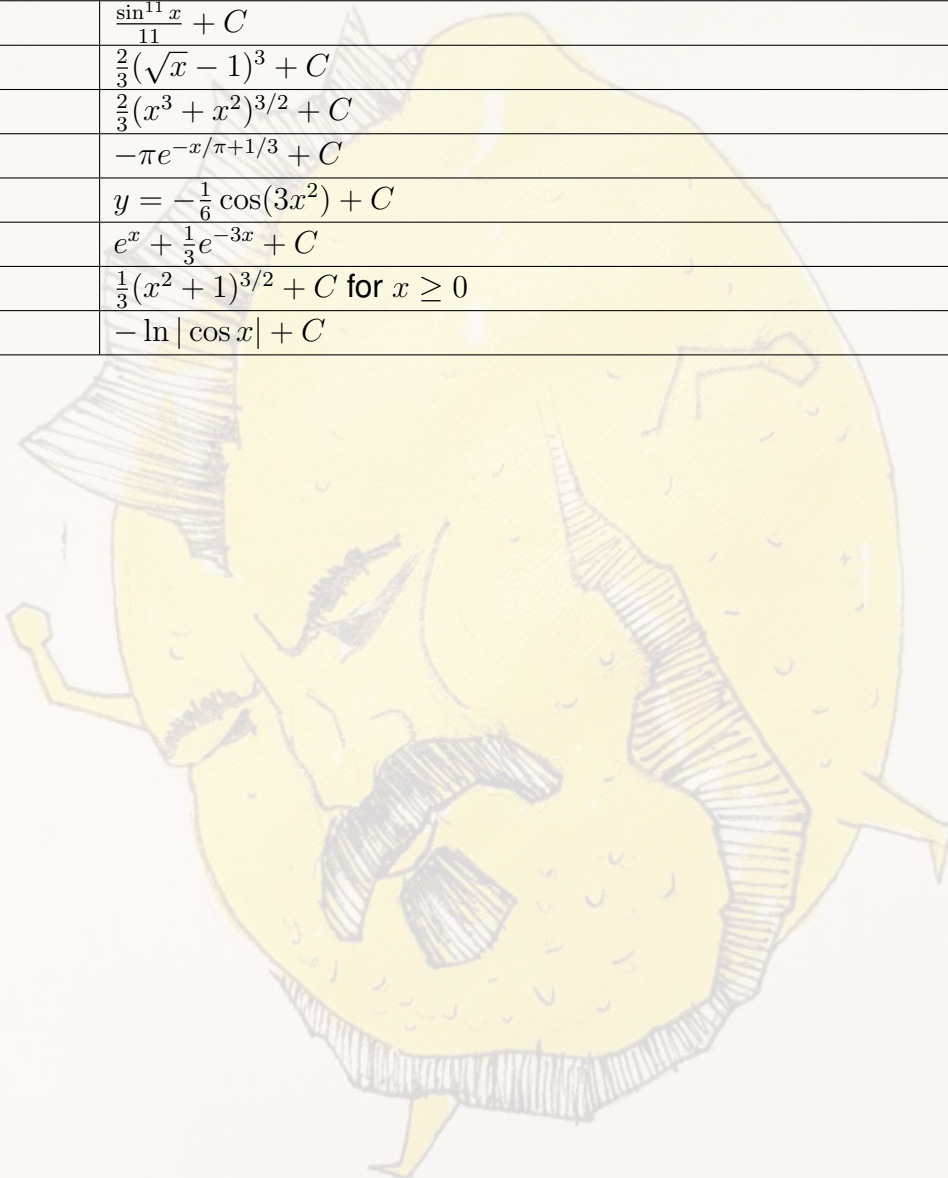
$$\boxed{-\ln |\cos x| + C}$$

or equivalently

$$\boxed{\ln |\sec x| + C}.$$

Answer Summary

Question	Answer
Warm Up a	$\frac{5}{8}x^8 + C$
Warm Up b	$\frac{16}{5}x^{5/2} - 2x - \frac{1}{x} + C$
Warm Up c	$-\frac{1}{2x^2} + \ln x + C$
Example 1	$\frac{(5x+4)^6}{30} + C$
Example 2	$\frac{(3x+2)^5}{15} + C$
Example 3	$\frac{(t^3+4)^6}{6} + C$
Example 4	$\frac{1}{6}(4x-5)^{3/2} + C$
Example 5	$\frac{2}{3}\sqrt{t^3+4} + C$
Example 6	$\frac{1}{2}\sin(2x+1) + C$
Example 7	$\frac{\sin^{11}x}{11} + C$
Example 8	$\frac{2}{3}(\sqrt{x}-1)^3 + C$
Example 9	$\frac{2}{3}(x^3+x^2)^{3/2} + C$
Example 10	$-\pi e^{-x/\pi+1/3} + C$
Example 11	$y = -\frac{1}{6}\cos(3x^2) + C$
Example 12	$e^x + \frac{1}{3}e^{-3x} + C$
Example 13	$\frac{1}{3}(x^2+1)^{3/2} + C$ for $x \geq 0$
Example 14	$-\ln \cos x + C$





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Mathematics made clear.