DP Analysis & Approaches (HL)

Maths with Lemon

HL Integration Review

Mathematician: ____

Warm Up

Find the gradient of the graph of

$$y = x + 2\cot x$$

at $x = \dfrac{\pi}{4}$.

Hint: Check Formula Booklet AHL 5.15.

Algebraic Manipulation Before Integration

Find

$$\int \frac{1 + 2\sin x}{\cos^2 x}\, dx.$$

HL Integral Practice

Worked Example	You Try
$\displaystyle\int \frac{2 + \sqrt{1 - x^2}}{\sqrt{1 - x^2}}\, dx$	$\displaystyle\int \frac{2\sqrt{1 - x^2} - 3}{\sqrt{1 - x^2}}\, dx$
$\displaystyle\int 2^x\, dx$	$\displaystyle\int 3^x\, dx$

$\displaystyle\int \frac{3}{\sqrt{1-x^2}}\,dx$	$\displaystyle\int \frac{4}{1+x^2}\,dx$
$\displaystyle\int \frac{2+3\sin x}{\cos^2 x}\,dx$	$\displaystyle\int \frac{3-\cos x}{\sin^2 x}\,dx$
$\displaystyle\int \sec x\,(2\sec x+3\tan x)\,dx$	$\displaystyle\int \csc x\,(5\csc x+2\cot x)\,dx$

Composition of Linear Functions

Find

$$\int 6\sec^2(3x+1)\,dx.$$

Composition of Linear Functions Practice

Worked Example	You Try
$\displaystyle\int 8\csc^2(2x-1)\,dx$	$\displaystyle\int 9\sec^2(3x+1)\,dx$

$\int 3^{2x}\,dx$	$\int 2^{5x}\,dx$
$\int \frac{3}{\sqrt{1-9x^2}}\,dx$	$\int \frac{10}{\sqrt{1-25x^2}}\,dx$
$\int 5\csc^2\left(\frac{x}{3}\right)dx$	$\int 3\sec^2\left(\frac{x}{4}\right)dx$
$\int \frac{1}{1+4x^2}\,dx$	$\int \frac{1}{1+16x^2}\,dx$

Integrals Resulting in $\arcsin$ and $\arctan$

Find

$$\int \frac{1}{x^2+2x+10}\,dx.$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)+C$$

Integrals Resulting in $\arcsin$ and $\arctan$ Practice

Worked Example	You Try
$\displaystyle\int \frac{1}{x^2+4x+5}\,dx$	$\displaystyle\int \frac{1}{x^2-6x+10}\,dx$
$\displaystyle\int \frac{1}{x^2+2x+10}\,dx$	$\displaystyle\int \frac{1}{x^2+4x+20}\,dx$
$\displaystyle\int \frac{1}{x^2+6x+11}\,dx$	$\displaystyle\int \frac{1}{x^2-10x+30}\,dx$
$\displaystyle\int \frac{1}{\sqrt{12-4x-x^2}}\,dx$	$\displaystyle\int \frac{1}{\sqrt{8+2x-x^2}}\,dx$
$\displaystyle\int \frac{1}{\sqrt{1+4x-x^2}}\,dx$	$\displaystyle\int \frac{1}{\sqrt{2-2x-x^2}}\,dx$

Integrals Requiring Partial Fraction Decomposition

Find

$$\int \frac{1}{x^2 - 5x + 6}\,dx.$$

Integrals Requiring Partial Fraction Decomposition Practice

Worked Examples	You Try
$\int \frac{2x+4}{x^2+4x+3}\,dx$	$\int \frac{x+1}{x^2+x-6}\,dx$
$\int \frac{4}{x^2-2x-3}\,dx$	$\int \frac{1}{x^2+5x+6}\,dx$
$\int \frac{3}{2x^2+x-1}\,dx$	$\int \frac{2}{3x^2+4x+1}\,dx$

Integration by Parts

Video explanation

The formula for integration by parts is

$$\boxed{\int u\,dv = uv - \int v\,du}.$$

The **tabular method** or **DI method** can be useful for repeated integration by parts. Select one part to differentiate and one part to integrate. Diagonal products appear outside the integral; horizontal products remain inside the integral.

Sign	Differentiate	Integrate
$+$	u	dv
$-$	u'	v
$+$	u''	$\int v\,dx$
$-$	u'''	$\iint v\,dx^2$

Integration by Parts Practice

Worked Example	You Try
$\int x\sin x\,dx$	$\int_1^e x^3\ln x\,dx$
$\int x^2e^x\,dx$	$\int e^x\sin x\,dx$
$\int x\cos 2x\,dx$	$\int x\cos 3x\,dx$

$\int x \sin\left(\frac{x}{2}\right) dx$	$\int x \sin\left(\frac{x}{3}\right) dx$
$\int x e^{-2x}\, dx$	$\int x e^{-3x}\, dx$
$\int x \ln x\, dx$	$\int x^2 \ln x\, dx$
$\int \frac{\ln x}{x^2}\, dx$	$\int \frac{\ln x}{x^3}\, dx$
$\int \sqrt{x} \ln x\, dx$	$\int \frac{\ln x}{\sqrt{x}}\, dx$

$\int x^2 e^{3x}\, dx$	$\int x^2 e^{-2x}\, dx$
$\int x^2 \sin 2x\, dx$	$\int x^2 \sin 3x\, dx$
$\int x^2 \cos\left(\frac{x}{3}\right) dx$	$\int x^2 \cos\left(\frac{x}{2}\right) dx$

Homogeneous Differential Equations

Video explanation

If a differential equation can be written as

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right),$$

then it is called a **homogeneous differential equation**. Use the substitution

$$y = vx, \qquad \frac{dy}{dx} = v + x\frac{dv}{dx},$$

to turn it into a separable differential equation in v and x.

Example. Solve

$$\frac{dy}{dx} = \frac{y+x}{x}, \qquad x > 0.$$

Homogeneous Differential Equation Practice

Worked Example	You Try
Find the general solution of $\dfrac{dy}{dx} = \dfrac{x^2 - y^2}{xy}$	Using $y = vx$, solve $\dfrac{dy}{dx} = \dfrac{y}{x} + \sqrt{\dfrac{y}{x}}$
$\dfrac{dy}{dx} = \dfrac{(2x+y)^2}{4x^2}$	$x^2\dfrac{dy}{dx} = 3x^2 - xy$
$xy' = x^2 \cos x + y$	$x^2y' = y^2 + xy + 4x^2$
$x^2\dfrac{dy}{dx} = y^2 + xy + 4x^2$	$y' = \dfrac{y}{x} + \dfrac{y^2}{x^2}$

Integrating Factors

Why it works   How it works

Differential equations of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

can be solved using an integrating factor

$$\boxed{\mu(x) = e^{\int P(x)\,dx}}.$$

Multiplying through by $\mu(x)$ gives

$$\frac{d}{dx}\big(\mu(x)y\big) = \mu(x)Q(x).$$

Integrating Factor Example. Solve

$$\frac{dy}{dx} + \frac{2y}{x} = x^3, \qquad x > 0.$$

Worked Example	You Try
Find the general solution of $y' + y = e^x$	$(x-1)y' = y - x^2 y$
$xy' + y = x^2 + 1$	$y' + y = \sin(e^x)$
$y' + xy = xe^{x^2}$	$x^2 y' + 2xy = \cos x$

Solutions

HL Integration Review

Warm Up

$$y' = 1 - 2\csc^2 x.$$

At $x = \pi/4$, $\csc^2(\pi/4) = 2$, so

$$\boxed{y' = -3}.$$

Algebraic Manipulation Before Integration

$$\frac{1 + 2\sin x}{\cos^2 x} = \sec^2 x + 2\tan x \sec x,$$

therefore

$$\boxed{\int \frac{1 + 2\sin x}{\cos^2 x}\,dx = \tan x + 2\sec x + C}.$$

HL Integral Practice

1.

$$\int \frac{2 + \sqrt{1 - x^2}}{\sqrt{1 - x^2}}dx = 2\int \frac{dx}{\sqrt{1 - x^2}} + \int dx = \boxed{2\arcsin x + x + C}.$$

2.

$$\int \frac{2\sqrt{1 - x^2} - 3}{\sqrt{1 - x^2}}dx = \boxed{2x - 3\arcsin x + C}.$$

3–4.

$$\boxed{\int 2^x dx = \frac{2^x}{\ln 2} + C}, \qquad \boxed{\int 3^x dx = \frac{3^x}{\ln 3} + C}.$$

5–6.

$$\boxed{\int \frac{3}{\sqrt{1 - x^2}}dx = 3\arcsin x + C}, \qquad \boxed{\int \frac{4}{1 + x^2}dx = 4\arctan x + C}.$$

7.

$$\int \frac{2 + 3\sin x}{\cos^2 x}dx = \int (2\sec^2 x + 3\tan x \sec x)dx = \boxed{2\tan x + 3\sec x + C}.$$

8.

$$\int \frac{3 - \cos x}{\sin^2 x}dx = \int (3\csc^2 x - \csc x \cot x)dx = \boxed{-3\cot x + \csc x + C}.$$

9–10.

$$\boxed{\int \sec x(2\sec x + 3\tan x)dx = 2\tan x + 3\sec x + C},$$

$$\boxed{\int \csc x(5\csc x + 2\cot x)dx = -5\cot x - 2\csc x + C}.$$

Composition of Linear Functions

$$\boxed{\int 6\sec^2(3x+1)dx = 2\tan(3x+1)+C}.$$

$$\boxed{\int 8\csc^2(2x-1)dx = -4\cot(2x-1)+C},$$

$$\boxed{\int 9\sec^2(3x+1)dx = 3\tan(3x+1)+C}.$$

$$\boxed{\int 3^{2x}dx = \frac{3^{2x}}{2\ln 3}+C}, \qquad \boxed{\int 2^{5x}dx = \frac{2^{5x}}{5\ln 2}+C}.$$

$$\boxed{\int \frac{3}{\sqrt{1-9x^2}}dx = \arcsin(3x)+C},$$

$$\boxed{\int \frac{10}{\sqrt{1-25x^2}}dx = 2\arcsin(5x)+C}.$$

$$\boxed{\int 5\csc^2(x/3)dx = -15\cot(x/3)+C},$$

$$\boxed{\int 3\sec^2(x/4)dx = 12\tan(x/4)+C}.$$

$$\boxed{\int \frac{dx}{1+4x^2} = \frac{1}{2}\arctan(2x)+C}, \qquad \boxed{\int \frac{dx}{1+16x^2} = \frac{1}{4}\arctan(4x)+C}.$$

Completing the Square: $\arctan$ and $\arcsin$

$$x^2+2x+10 = (x+1)^2+9,$$

so

$$\boxed{\int \frac{dx}{x^2+2x+10} = \frac{1}{3}\arctan\left(\frac{x+1}{3}\right)+C}.$$

Practice answers:

$$\boxed{\int \frac{dx}{x^2+4x+5} = \arctan(x+2)+C}$$

$$\boxed{\int \frac{dx}{x^2-6x+10} = \arctan(x-3)+C}$$

$$\boxed{\int \frac{dx}{x^2+4x+20} = \frac{1}{4}\arctan\left(\frac{x+2}{4}\right)+C}$$

$$\boxed{\int \frac{dx}{x^2+6x+11} = \frac{1}{\sqrt{2}}\arctan\left(\frac{x+3}{\sqrt{2}}\right)+C}$$

$$\boxed{\int \frac{dx}{x^2-10x+30} = \frac{1}{\sqrt{5}}\arctan\left(\frac{x-5}{\sqrt{5}}\right)+C}.$$

For the square-root forms, complete the square:

$$12 - 4x - x^2 = 16 - (x+2)^2,$$

$$\boxed{\int \frac{dx}{\sqrt{12 - 4x - x^2}} = \arcsin\left(\frac{x+2}{4}\right) + C}.$$

$$8 + 2x - x^2 = 9 - (x-1)^2,$$

$$\boxed{\int \frac{dx}{\sqrt{8 + 2x - x^2}} = \arcsin\left(\frac{x-1}{3}\right) + C}.$$

$$1 + 4x - x^2 = 5 - (x-2)^2,$$

$$\boxed{\int \frac{dx}{\sqrt{1 + 4x - x^2}} = \arcsin\left(\frac{x-2}{\sqrt{5}}\right) + C}.$$

$$2 - 2x - x^2 = 3 - (x+1)^2,$$

$$\boxed{\int \frac{dx}{\sqrt{2 - 2x - x^2}} = \arcsin\left(\frac{x+1}{\sqrt{3}}\right) + C}.$$

Partial Fractions

$$\frac{1}{(x-2)(x-3)} = -\frac{1}{x-2} + \frac{1}{x-3},$$

so

$$\boxed{\int \frac{dx}{x^2 - 5x + 6} = \ln\left|\frac{x-3}{x-2}\right| + C}.$$

Practice:

$$\frac{2x+4}{(x+1)(x+3)} = \frac{1}{x+1} + \frac{1}{x+3}$$

$$\boxed{\ln|x+1| + \ln|x+3| + C}.$$

$$\frac{x+1}{(x+3)(x-2)} = \frac{2/5}{x+3} + \frac{3/5}{x-2}$$

$$\boxed{\frac{2}{5}\ln|x+3| + \frac{3}{5}\ln|x-2| + C}.$$

$$\frac{4}{(x-3)(x+1)} = \frac{1}{x-3} - \frac{1}{x+1}$$

$$\boxed{\ln\left|\frac{x-3}{x+1}\right| + C}.$$

$$\boxed{\int \frac{dx}{(x+2)(x+3)} = \ln\left|\frac{x+2}{x+3}\right| + C}.$$

$$\frac{3}{(2x-1)(x+1)} = \frac{2}{2x-1} - \frac{1}{x+1}$$

$$\boxed{\ln|2x-1| - \ln|x+1| + C}.$$

$$\frac{2}{(3x+1)(x+1)} = \frac{3}{3x+1} - \frac{1}{x+1}$$

$$\boxed{\ln|3x+1| - \ln|x+1| + C}.$$

Integration by Parts

1.

$$\int x \sin x \, dx = -x\cos x + \sin x + C.$$

$$\boxed{-x\cos x + \sin x + C}$$

2.

$$\int_1^e x^3 \ln x \, dx = \left[\frac{x^4}{4}\ln x - \frac{x^4}{16}\right]_1^e = \boxed{\frac{3e^4+1}{16}}.$$

3.

$$\boxed{\int x^2 e^x dx = e^x(x^2 - 2x + 2) + C}.$$

4. Let $I = \int e^x \sin x \, dx$. Integrating by parts twice gives

$$2I = e^x(\sin x - \cos x),$$

so

$$\boxed{I = \frac{e^x}{2}(\sin x - \cos x) + C}.$$

5–6.

$$\boxed{\int x \cos 2x \, dx = \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C},$$

$$\boxed{\int x \cos 3x \, dx = \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x + C}.$$

7–8.

$$\boxed{\int x \sin(x/2) dx = -2x\cos(x/2) + 4\sin(x/2) + C},$$

$$\boxed{\int x \sin(x/3) dx = -3x\cos(x/3) + 9\sin(x/3) + C}.$$

9–10.

$$\boxed{\int xe^{-2x} dx = -e^{-2x}\left(\frac{x}{2} + \frac{1}{4}\right) + C},$$

$$\boxed{\int xe^{-3x} dx = -e^{-3x}\left(\frac{x}{3} + \frac{1}{9}\right) + C}.$$

11–15.

$$\boxed{\int x \ln x \, dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C},$$

$$\boxed{\int x^2 \ln x \, dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C},$$

$$\boxed{\int \frac{\ln x}{x^2} dx = -\frac{\ln x + 1}{x} + C},$$

$$\boxed{\int \frac{\ln x}{x^3} dx = -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C},$$

$$\boxed{\int \sqrt{x} \ln x \, dx = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C},$$

$$\boxed{\int \frac{\ln x}{\sqrt{x}} dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C}.$$

Repeated integration by parts:

$$\boxed{\int x^2 e^{3x} dx = e^{3x}\left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27}\right) + C},$$

$$\boxed{\int x^2 e^{-2x} dx = -e^{-2x}\left(\frac{x^2}{2} + \frac{x}{2} + \frac{1}{4}\right) + C},$$

$$\boxed{\int x^2 \sin 2x\, dx = -\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C},$$

$$\boxed{\int x^2 \sin 3x\, dx = -\frac{x^2}{3}\cos 3x + \frac{2x}{9}\sin 3x + \frac{2}{27}\cos 3x + C},$$

$$\boxed{\int x^2 \cos(x/3) dx = 3x^2 \sin(x/3) + 18x\cos(x/3) - 54\sin(x/3) + C},$$

$$\boxed{\int x^2 \cos(x/2) dx = 2x^2 \sin(x/2) + 8x\cos(x/2) - 16\sin(x/2) + C}.$$

Homogeneous Differential Equations

Introductory example:

$$\frac{dy}{dx} = \frac{y+x}{x} = \frac{y}{x} + 1.$$

Set $y = vx$. Then $y' = v + xv'$, hence

$$v + xv' = v + 1 \Rightarrow xv' = 1.$$

Thus $v = \ln x + C$, and

$$\boxed{y = x\ln x + Cx}.$$

1. For $y' = \dfrac{x^2 - y^2}{xy}$, set $y = vx$:

$$v + xv' = \frac{1 - v^2}{v} = \frac{1}{v} - v.$$

Hence

$$xv' = \frac{1}{v} - 2v = \frac{1 - 2v^2}{v}, \qquad \frac{v}{1 - 2v^2} dv = \frac{dx}{x}.$$

Therefore

$$-\frac{1}{4}\ln|1 - 2v^2| = \ln|x| + C,$$

which can be written as

$$\boxed{x^4 - 2x^2y^2 = C}.$$

2. For $y' = y/x + \sqrt{y/x}$, let $y = vx$:

$$v + xv' = v + \sqrt{v} \Rightarrow xv' = \sqrt{v}.$$

Thus

$$2\sqrt{v} = \ln|x| + C,$$

and

$$\boxed{y = \frac{x}{4}(\ln|x| + C)^2}.$$

3. For $y' = \dfrac{(2x+y)^2}{4x^2}$:

$$v + xv' = \frac{(2+v)^2}{4},$$

so

$$xv' = 1 + \frac{v^2}{4}.$$

Hence

$$\int \frac{4\,dv}{v^2+4} = \int \frac{dx}{x},$$

therefore

$$\boxed{2\arctan\left(\frac{y}{2x}\right) = \ln|x| + C}.$$

4. For $x^2y' = 3x^2 - xy$:

$$y' + \frac{y}{x} = 3.$$

This is linear (and also reducible using $y = vx$). With $y = vx$:

$$v + xv' + v = 3 \Rightarrow xv' = 3 - 2v.$$

Thus

$$\boxed{y = \frac{3}{2}x + \frac{C}{x}}.$$

5. For $xy' = x^2 \cos x + y$:

$$y' - \frac{y}{x} = x \cos x.$$

Using $y = vx$, $xv' = x \cos x$, so $v' = \cos x$. Hence

$$\boxed{y = x \sin x + Cx}.$$

6. For $x^2y' = y^2 + xy + 4x^2$, set $y = vx$:

$$v + xv' = v^2 + v + 4 \Rightarrow xv' = v^2 + 4.$$

Thus

$$\frac{1}{2}\arctan\left(\frac{v}{2}\right) = \ln|x| + C,$$

so

$$\boxed{\arctan\left(\frac{y}{2x}\right) = 2\ln|x| + C}.$$

The final practice equation $y' = y/x + y^2/x^2$ gives

$$v + xv' = v + v^2 \Rightarrow xv' = v^2,$$

therefore

$$-\frac{1}{v} = \ln|x| + C,$$

and

$$\boxed{y = -\frac{x}{\ln|x| + C}}.$$

Integrating Factors

Introductory example:

$$y' + \frac{2}{x}y = x^3, \qquad \mu = e^{\int 2/x\,dx} = x^2.$$

Then

$$(x^2y)' = x^5,$$

so

$$\boxed{y = \frac{x^4}{6} + \frac{C}{x^2}}.$$

1. $y' + y = e^x$. Here $\mu = e^x$:

$$(e^x y)' = e^{2x} \Rightarrow \boxed{y = \frac{1}{2}e^x + Ce^{-x}}.$$

2. $(x-1)y' = y - x^2y$:

$$y' - \frac{1 - x^2}{x - 1}y = 0.$$

Since $\dfrac{1-x^2}{x-1} = -(x+1)$, this becomes

$$y' + (x+1)y = 0.$$

Thus

$$\boxed{y = Ce^{-x^2/2-x}}.$$

3. $xy' + y = x^2 + 1$:

$$y' + \frac{1}{x}y = x + \frac{1}{x}, \qquad \mu = x.$$

Hence

$$(xy)' = x^2 + 1,$$

so

$$\boxed{y = \frac{x^2}{3} + 1 + \frac{C}{x}}.$$

4. $y' + y = \sin(e^x)$:

$$(e^x y)' = e^x \sin(e^x).$$

Let $u = e^x$, so

$$e^x y = -\cos(e^x) + C,$$

and

$$\boxed{y = e^{-x}\big(C - \cos(e^x)\big)}.$$

5. $y' + xy = xe^{x^2}$:

$$\mu = e^{x^2/2}, \qquad (e^{x^2/2}y)' = xe^{3x^2/2}.$$

Therefore

$$\boxed{y = \frac{1}{3}e^{x^2} + Ce^{-x^2/2}}.$$

6. $x^2y' + 2xy = \cos x$:

$$y' + \frac{2}{x}y = \frac{\cos x}{x^2}, \qquad \mu = x^2.$$

Then

$$(x^2y)' = \cos x,$$

so

$$\boxed{y = \frac{\sin x + C}{x^2}}.$$

Maths with Lemon

Mathematics made clear.