Lesson 10.2 – Worksheet

First-Order Differential Equations and Euler's Method

Mathematician: _____ **Class:** _____

A. Terminology and modelling

1. Classify each as first- or second-order, and linear or nonlinear: (a) $y' = x + y$; (b) $(y')^2 + yy' = y^2 - 4$; (c) $y'' = 5x + 2y + e^x y'$.

2. Write differential equations for (a) water leaving a bucket at a rate proportional to $\sqrt{V}$; (b) bacterial growth proportional to B; (c) height growth inversely proportional to height H; (d) meteorite acceleration inversely proportional to the square of distance s; (e) disease spread proportional to $I(N - I)$; (f) rumour spread proportional to $R(N - R)$ and inversely proportional to t. Define any constants.

3. Solve $\frac{dy}{dx} = e^{2x}$ given $y(0) = 3$. State both general and particular solutions.

B. Euler's method by hand

4. Use $h = 0.2$ to estimate $y(1)$ if $y' = x + 2y$ and $y(0) = 2$. Show a table with x_n, y_n and the gradient.

5. Use $h = 0.2$ to estimate $y(1)$ if $y' = xy$ and $y(0) = 1$.

6. Use $h = 0.2$ and $y(0) = 1$ to estimate $y(1)$ for each: (a) $y' = x + y$; (b) $y' = x - y$; (c) $y' = x^2 + y^2$; (d) $y' = ye^x$. Record at least five iterations for each.

C. Numerical applications

7. Speeds at $t = 0, 3, 6, 9, 12, 15$ seconds are $0, 6, 12, 19, 24, 27$ m/s. Use forward Euler to estimate distance in the first 15 seconds.

8. Repeat for speeds $20, 15, 10, 8, 6, 5$ m/s at the same times. Explain why the final speed is not used.

9. For $y' = y$, $y(0) = 1$, estimate $y(1)$ using (a) $h = 0.5$ and (b) $h = 0.25$. Compare with e.

D. Slope fields and solution curves

10. Sketch a slope field for $y' = y$ at integer grid points $-2 \leq x, y \leq 2$. Identify all points where the slope is zero.

11. Sketch a slope field for $y' = -xy$ on the same grid. Identify the zero-slope loci and determine the sign of the gradient in each quadrant.

12. A slope field has gradients that depend only on x, are negative when $x < 0$, and positive when $x > 0$. Which is compatible: $y' = 1/x$, $y' = y$, $y' = x$, or $y' = -1/y$? Justify.

13. For $y' = x - y$, find the gradient at $(1, 0)$, $(0, 1)$ and $(-1, 0)$. Use these to sketch short tangent segments and a possible solution through $(0, 1)$.

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Analytical Solutions

1. Classification

(a) First-order, linear: only y' and y appear to the first power. (b) First-order, nonlinear: $(y')^2$, yy' and y^2 occur. (c) Second-order, linear: the highest derivative is y'' and the equation is linear in y, y', y''.

2. Mathematical models

Let $k > 0$ denote a proportionality constant. (a) $dV/dt = -k\sqrt{V}$ (loss of volume). (b) $dB/dt = kB$. (c) $dH/dt = k/H$. (d) $d^2s/dt^2 = -k/s^2$ when acceleration points toward the planet (the sign depends on coordinate orientation). (e) $dI/dt = kI(N - I)$. (f) $dR/dt = kR(N - R)/t$, for $t > 0$. **3. Exact integration**

$$y = \int e^{2x}\,dx = \frac{1}{2}e^{2x} + C.$$

$y(0) = 3$ gives $C = 5/2$, hence $\boxed{y = \frac{1}{2}e^{2x} + \frac{5}{2}}$. **4–6. Euler iterations**

For all questions use $x_{n+1} = x_n + 0.2$ and $y_{n+1} = y_n + 0.2f(x_n, y_n)$. The table shows y_n at $x_n = 0, 0.2, \ldots, 1$; entries are rounded for display, but calculations use unrounded values.

Problem	y_0	y_1	y_2	y_3	y_4	y_5
4: $x + 2y$	2.000000	2.800000	3.960000	5.624000	7.993600	11.351040
5: xy	1.000000	1.000000	1.040000	1.123200	1.257984	1.459261
6(a): $x + y$	1.000000	1.200000	1.480000	1.856000	2.347200	2.976640
6(b): $x - y$	1.000000	0.800000	0.680000	0.624000	0.619200	0.655360
6(c): $x^2 + y^2$	1.000000	1.200000	1.496000	1.975603	2.828205	4.555953
6(d): ye^x	1.000000	1.200000	1.493137	1.938636	2.645121	3.822487

For example, in problem 4: $y_1 = 2 + 0.2(0 + 2 \cdot 2) = 2.8$ and $y_2 = 2.8 + 0.2(0.2 + 2 \cdot 2.8) = 3.96$. In problem 5: $y_1 = 1 + 0.2(0 \cdot 1) = 1$ and $y_2 = 1 + 0.2(0.2 \cdot 1) = 1.04$. **7. Accelerating car**

$\Delta s \approx \sum_{n=0}^{4} v_n \Delta t = 3(0 + 6 + 12 + 19 + 24) = \boxed{183\text{ m}}$. **8. Slowing car**

$\Delta s \approx 3(20 + 15 + 10 + 8 + 6) = \boxed{177\text{ m}}$. Forward Euler uses the speed at the start of each of the five intervals; the speed at 15 s is the endpoint value. **9. Step-size comparison**

For $y' = y$, the recurrence is $y_{n+1} = (1 + h)y_n$, so $y(1) \approx (1 + h)^{1/h}$. With $h = 0.5$: $1.5^2 = \boxed{2.25}$. With $h = 0.25$: $1.25^4 = \boxed{2.44140625}$. The exact value is $e \approx 2.718282$. Both underestimate, and the smaller step gives a closer approximation. **10. Slope field $y' = y$**

At every grid point (x, y) draw slope $m = y$, independent of x. For $y = -2, -1, 0, 1, 2$ the slopes are respectively $-2, -1, 0, 1, 2$. All points on $y = 0$ have zero slope. Solutions are $y = Ce^x$. **11. Slope field $y' = -xy$**

At (x, y) the gradient is $-xy$. It is zero on $x = 0$ and $y = 0$, negative in quadrants I and III, positive in II and IV. For instance at $(1, 1)$ the slope is -1, and at $(-1, 1)$ it is $+1$. Solutions have the form $y = Ce^{-x^2/2}$. **12. Matching slope patterns**

$\boxed{y' = x}$. It is independent of y, negative for $x < 0$, zero at $x = 0$, positive for $x > 0$. The option $1/x$ also changes sign but is undefined at $x = 0$ and is not zero there; the stated smooth zero-slope transition identifies $y' = x$. **13. Local gradients and solution**

For $y' = x - y$: at $(1, 0)$, $m = 1$; at $(0, 1)$, $m = -1$; at $(-1, 0)$, $m = -1$. Solving $y' + y = x$ using integrating factor e^x gives $y = x - 1 + Ce^{-x}$. Since $y(0) = 1$, $C = 2$, so the curve is $\boxed{y = x - 1 + 2e^{-x}}$. Its tangent at $(0, 1)$ has slope -1.



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